A Lomax-inverse Lindley Distribution: Model, Properties and Applications to Lifetime Data

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This article proposed a new extension of the Inverse Lindley distribution called "Lomax-Inverse Lindley distribution" which is more flexible compared to the Inverse Lindley distribution and other similar models. The paper derives and discusses some Statistical properties of the new distribution which include the limiting behavior, quantile function, reliability functions and distribution of order statistics. The parameters of the new model are estimated by method of maximum likelihood estimation. Conclusively, three lifetime datasets were used to evaluate the usefulness of the proposed model and the results indicate that the proposed extension is more flexible and performs better than the other distributions considered in this study.

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1 Introduction

Prof. D. V. Lindley investigated a probability distribution in context of fiducial statistic as a counter example of Bayesian theory which was later called "Lindley distribution" (Lindley [1]). The fundamental properties of the Lindley distribution (LIND) with application to waiting time data was discussed by Ghitany et al. [2]. Since then, many researchers have studied this distribution, for instance, Mazucheli and Achcar [3] worked on the Lindley distribution applied to competing risks lifetime data. Krishna and Kumar [4] estimated the parameter of Lindley distribution with progressive Type-II censoring scheme and also showed that it may be better lifetime model than exponential, lognormal and gamma distributions in some real life situations. Singh and Gupta [5] used the distribution under load sharing system models. Al-Mutairi et al. [6] developed an inferential procedure of the stress-strength parameter when both stress and strength variables follow Lindley distribution and discovered that the Lindley distribution is useful when the data show increasing failure rate. This particular property makes the use of Lindley distribution in lifetime data analysis more frequent than the exponential distribution. Despite the important properties and various applications of the Lindley distribution in many disciplines, its applicability may be restricted to non-monotone hazard rate data (see Sharma et al. [7]). To solve the above mentioned problem therefore, several extensions of the Lindley distribution have been proposed in the literature and some of the recent generalizations are summarized as follows: Zakerzadeh and Dolati [8] and Nadarajah et al. [9] introduced a three parameters extension of the Lindley distribution called "a generalized Lindley distribution". Ghitany et al. [10] proposed two parameter generalizations of the Lindley distribution, called the power Lindley distribution which was generated using the power transformations to the Lindley distribution. Merovci [11] investigated transmuted Lindley distribution and Merovci and Elbatal [12] studied the transmuted Lindley-geometric distribution. The beta-Lindley distribution was also introduced by Merovci and Sharma [13]. Elbatal and Elgarhy [14] studied the statistical and mathematical properties of Kumaraswamy Quasi-Lindley distribution and Kumaraswamy Lindley distribution was proposed and discussed by Akmakyapan and Kadlar [15]. The exponentiated power Lindley distribution has been introduced by Ashour and Eltehiwy [16] and the generalized weighted Lindley distribution by Ramos and Louzada [17].

The inverse Lindley distribution is a two component mixture of inverse exponential distribution and special case of inverse gamma distribution. From the introduction above, it can be seen that authors mainly focused on the Lindley distribution and little has been said about the inverse Lindley distribution. Sharma et al. [18] discussed the properties of inverse Lindley distribution with application to stress strength reliability analysis. Sharma et al. [19] introduced a two parameter extension of inverse Lindley distribution (generalized inverse Lindley distribution). Also, Alkarni [20] proposed a three parameter inverse Lindley distribution (extended inverse Lindley distribution) with application to maximum flood level data and the inverse weighted Lindley distribution was introduced and studied by Ramos et al. [21].

Recently, different families of distributions have been introduced, studied and used in the literature for instance there is Odd Lindley-G family by Gomes-Silva et al., [22], Lindley-G family by Cakmakyapan and Ozel [23], a new Weibull-G family by Tahir *et al.* [24] and a Lomax-G family by Cordeiro et al. [25] etc. Also, recent literature review has shown that using Lomax generator of probability distributions (Lomax-G family) by Cordeiro et al., [25] to add two parameters to some classical continuous distributions has led to compound distributions with greater degree of skewness and flexibility for modeling real life datasets (Ieren and Kuhe [26], Omale et al. [27], Venegas et al., [28], Ieren et al., [29]).

Based on the recorded advantage of the Lomax-G family above and the desire to increase the flexibility of the inverse Lindley distribution, this this study seeks to extend the Inverse Lindley distribution (INLIND) using the Lomax generator of probability distributions (Lomax-G family) proposed by Cordeiro et al. [25] and hope that it will produce a better model for analyzing real life situations in various fields of study especially reliability analysis.

The cumulative distribution function (c.d.f) and probability density function (pdf) of the Inverse Lindley distribution (INLIND) are defined as:

$$G(x) = \left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}} \tag{1}$$

and

$$g(x) = \frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}} \tag{2}$$

respectively, for $x > 0$ and $\theta > 0$ where θ is the scale parameter of INLIND.

This article is divided in sections as follows: the new distribution is derived with its validity check and graphical representation in section 2. Section 3 derived some properties of the proposed distribution. The estimation of unknown parameters of the distribution using maximum likelihood estimation is provided in section 4. An application of the proposed distribution to three real life datasets is done in section 5 and the summary and conclusion of the study is given in section 6.

2 The Lomax Inverse Lindley Distribution (LOMINLIND)

2.1 Definition

According to Cordeiro et al*.,* [25], the cumulative distribution function (*cdf*) of the Lomax-G family of distributions is defined as:

$$F(x) = \int_0^{-\log[1-G(x)]} \alpha\beta^\alpha \frac{dt}{(\beta+t)^{\alpha+1}} \tag{3}$$

"where $G(x)$ is the cdf of any continuous distribution to be modified or generalized and $\alpha > 0$ and $\beta > 0$ are the two extra shape parameters of the Lomax-G family".

Solving equation (3) above and evaluating the integrand in the equation gives:

$$F(x) = 1 - \left\{\frac{\beta}{\beta - \log[1-G(x)]}\right\}^\alpha \tag{4}$$

Hence, equation (4) is the simplified cumulative distribution function of the Lomax-G family of distributions proposed by Cordeiro et al*.,* [25] and the corresponding probability density function of the family can be obtained from equation (4) by taking the derivative of the cdf, $F(x)$ with respect to *x* and is obtained as:

$$f(x) = \alpha\beta^\alpha \frac{g(x)}{[1-G(x)]\{\beta - \log[1-G(x)]\}^{\alpha+1}} \tag{5}$$

"where $g(x)$ and $G(x)$ represent the *pdf* and the *cdf* of any continuous distribution to be extended respectively".

Making substitution of equation (1) and (2) in (4) and (5) above and simplifying, we obtain the *cdf* and *pdf* of the LOMINLIND for a random variable *X* as:

$$F(x) = 1 - \left\{ \frac{\beta}{\beta - \log\left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]} \right\}^{\alpha} = 1 - \beta^{\alpha}\left\{\beta - \log\left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}^{-\alpha} \quad (6)$$

and

$$f(x) = \frac{\alpha\beta^{\alpha}\frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left(1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right)^{-1}}{\left\{\beta - \log\left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}^{\alpha+1}} \quad (7)$$

respectively, for $x > 0; \alpha, \beta, \theta > 0$.

2.2 Validity of the model *f(x)*

In probability theory, a continuous probability distribution is said to be valid if and only if the following integral in equation (8) is true, that is

$$\int_{-\infty}^{\infty} f(x)dx = 1 \qquad (8)$$

Proof

Considering the *pdf* of the LOMINLIND, which is given as

$$f(x) = \frac{\alpha\beta^{\alpha}\frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left(1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right)^{-1}}{\left\{\beta - \log\left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}^{\alpha+1}}$$

Substituting this pdf in equation (8) above and simplifying, we have

$$\int_{0}^{\infty} f(x)dx = \int_{0}^{\infty}\left(\frac{\alpha\beta^{\alpha}\frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left(1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right)^{-1}}{\left\{\beta - \log\left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}^{\alpha+1}}\right)dx$$

$$\int_0^\infty f(x)dx = \int_0^\infty \left(\frac{\alpha\beta^\alpha \frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\beta^{-(\alpha+1)}\left\{1-\beta^{-1}\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}^{-(\alpha+1)}}{\left(1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right)} \right) dx \qquad (9)$$

Now, from equation (9), let

$$y = 1 - \beta^{-1}\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]$$

Such that

$$\frac{dy}{dx} = \frac{\frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\beta\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]}$$

Which implies that

$$dx = -\frac{\beta\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]dy}{\frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}$$

Substituting for y and dx in (9) and simplifying, we have the following results:

$$\int_0^\infty f(x)dx = \int_0^\infty \alpha y^{-(\alpha+1)}dy \qquad (10)$$

Integrating and applying the limit in equation (10) above results in the following:

$$\int_0^\infty f(x)dx = \int_0^\infty \alpha y^{-(\alpha+1)}dy = -\left[y^{-\alpha}\right]_0^\infty \qquad (11)$$

But recall that

$$y = 1 - \beta^{-1} \log \left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right]$$

Hence, substituting for y in equation (11) and simplifying will result in the following:

$$\int_0^\infty f(x)dx = -\left[y^{-\alpha} \right]_0^\infty = -\left\{ \lim_{x \to \infty} \left(1 - \beta^{-1} \log \left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right)^{-\alpha} - \lim_{x \to 0} \left(1 - \beta^{-1} \log \left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right)^{-\alpha} \right\} \quad (12)$$

Recall that $\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}}$ is the cdf of the Inverse Lindley distribution (INLIND) and its limit as X

approaches infinity, $x \to \infty$ is equal to one (1) while its limit as X tends zero, $x \to 0$ is equal to zero (0). Therefore, from equation (12), we have:

$$\int_0^\infty f(x)dx = -\left\{ \left(1 - \beta^{-1} \log \left[1 - (1) \right] \right)^{-\alpha} - \left(1 - \beta^{-1} \log \left[1 - (0) \right] \right)^{-\alpha} \right\}$$

$$\int_0^\infty f(x)dx = -\left\{ \left(1 - \beta^{-1} \log[0] \right)^{-\alpha} - \left(1 - \beta^{-1} \log[1] \right)^{-\alpha} \right\} = -\left\{ 0 - (1-0)^{-\alpha} \right\} = -\left\{ 0 - (1)^{-\alpha} \right\} = 0 + 1 = 1$$
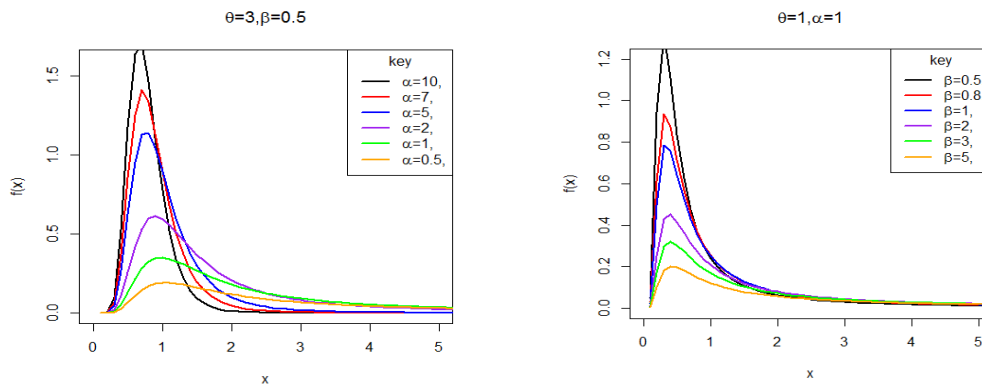
Therefore,

$$\int_0^\infty f(x)dx = 1$$

and hence the proposed pdf of the LOMINLIND in equation (7) is a valid pdf.

2.3 Graphical representation of the Pdf and Cdf of LOMINLIND

The pdf and cdf of the LOMINLIND using some arbitrary parameter values are shown in Figs. 1 and 2.
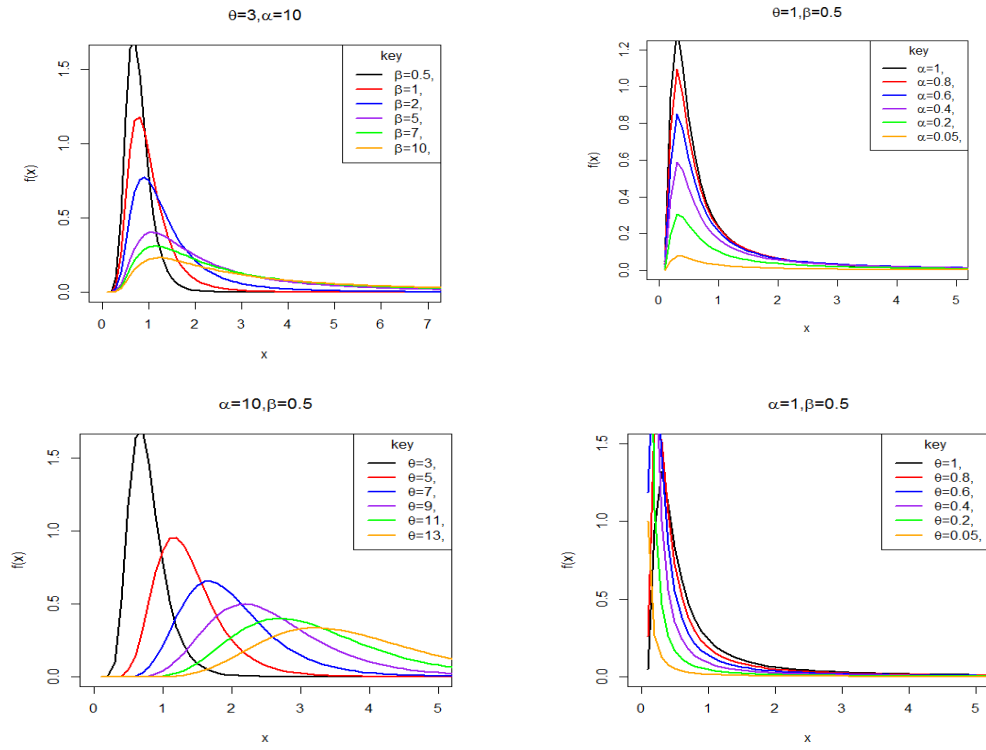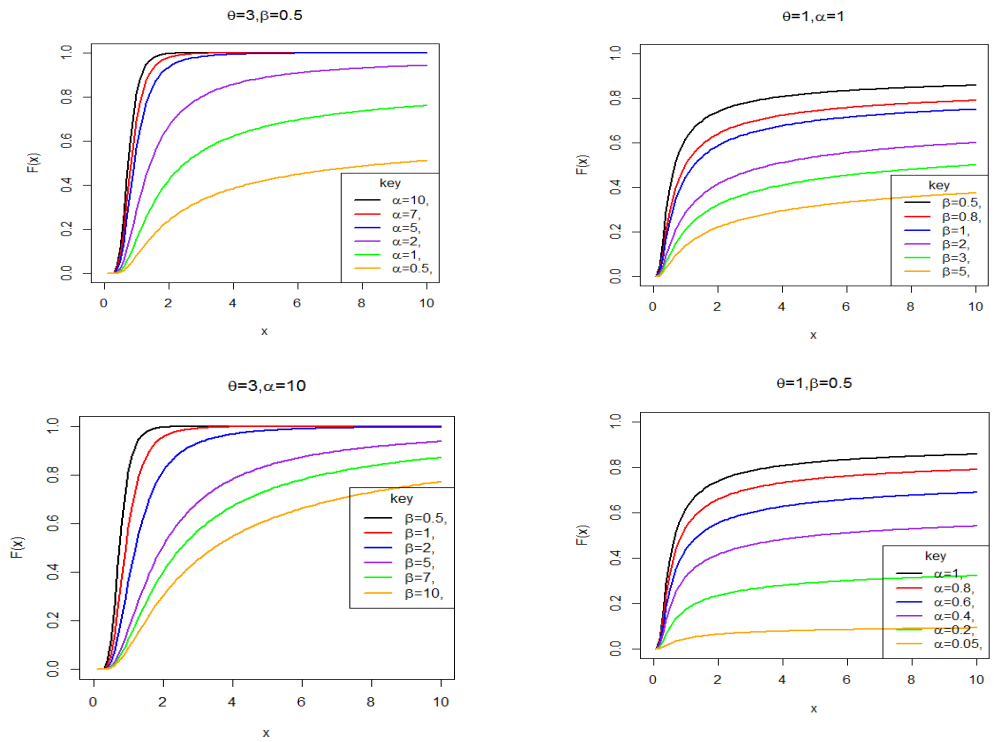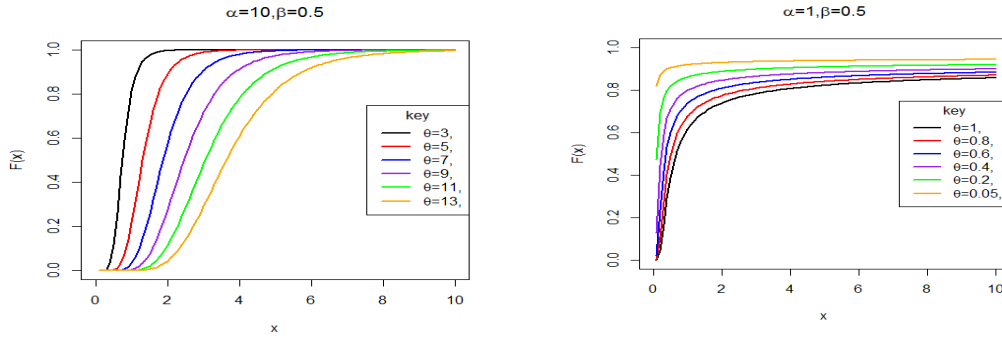
Fig. 1. PDF of LOMINLIND

Fig. 2. CDF of LOMINLIND

3 Mathematical and Statistical Properties of LOMINLIND

3.1 Limiting behavior

Under this subsection, the limiting behavior of the LOMINLIND is being investigated as follows:

Lemma 1: The limit of the pdf of the LOMINLIND as X approaches infinity, $x \to \infty$ is equal to zero (0) and its limit as X tends to zero (0), $x \to 0$ is also equal to zero (0).

Proof

(i) The limit of the pdf of the LOMINLIND, f(x) as X approaches infinity, $x \to \infty$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left\{ \frac{\alpha \beta^\alpha \dfrac{\theta^2}{\theta+1} \left(\dfrac{1+x}{x^3} \right) e^{-\frac{\theta}{x}} \left(1 - \left(\left(1 + \dfrac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right)^{-1}}{\left\{ \beta - \log \left[1 - \left(\left(1 + \dfrac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right\}^{\alpha+1}} \right\} \tag{13}$$

Recall that $\left(1 + \dfrac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}}$ is the cdf of the INLIND and its limit as X approaches infinity, $x \to \infty$ is equal to one (1) and that $\dfrac{\theta^2}{\theta+1} \left(\dfrac{1+x}{x^3} \right) e^{-\frac{\theta}{x}}$ is the pdf of the INLIND and its limit as X approaches infinity, $x \to \infty$ is equal to zero (0), therefore simplifying equation (13) above gives:

$$\lim_{x \to \infty} f(x) = (0) \frac{\alpha \beta^\alpha \left(1 - (1) \right)^{-1}}{\left\{ \beta - \log \left[1 - (1) \right] \right\}^{\alpha+1}} = 0 \tag{14}$$

(ii) The limit of the pdf of the LOMINLIND, f(x) as X tends to zero (0), $x \to 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left\{ \frac{\alpha \beta^{\alpha} \frac{\theta^2}{\theta+1} \left(\frac{1+x}{x^3} \right) e^{-\frac{\theta}{x}} \left(1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right)^{-1}}{\left\{ \beta - \log \left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right\}^{\alpha+1}} \right\} \qquad (15)$$

Recall that $\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}}$ is the cdf of the INLIND and its limit as X tends to zero (0), $x \to 0$ is

equal to zero (0) and also that $\frac{\theta^2}{\theta+1} \left(\frac{1+x}{x^3} \right) e^{-\frac{\theta}{x}}$ is the pdf of the INLIND and its limit as X tends to zero

(0), $x \to 0$ is equal to zero (0), therefore simplifying the equation (15) above gives:

$$\lim_{x \to 0} f(x) = (0) \frac{\alpha \beta^{\alpha} \left(1 - (0) \right)^{-1}}{\left\{ \beta - \log \left[1 - (0) \right] \right\}^{\alpha+1}} = 0 \qquad (16)$$

Lemma 2: The limit of the cdf of the LOMINLIND, F(x) as X approaches infinity, $x \to \infty$ is equal to one (1) and limit of the cdf of the LOMINLIND, F(x) as X tends to zero (0), $x \to 0$ is equal to zero (0).

Proof:

(i) The limit of the cdf of LOMINLIND, F(x) as X approaches infinity, $x \to \infty$

$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty} \left\{ 1 - \beta^{\alpha} \left\{ \beta - \log \left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right\}^{-\alpha} \right\} \qquad (17)$$

$$\lim_{x \to \infty} F(x) = 1 - \left\{ \frac{b}{b - \log \left[1 - 1 \right]} \right\}^{a} = 1 - \left\{ \frac{b}{b - \log(0)} \right\}^{a} = 1 \qquad (18)$$

(ii) The limit of the cdf of LOMINLIND, F(x) as X tends to zero (0), $x \to 0$

$$\lim_{x \to 0} F(x) = \lim_{x \to 0} \left\{ 1 - \beta^{\alpha} \left\{ \beta - \log \left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right\}^{-\alpha} \right\} \qquad (19)$$

$$\lim_{x \to 0} F(x) = 1 - \left\{ \frac{b}{b - \log \left[1 - (0) \right]} \right\}^{a} = 1 - \left\{ \frac{b}{b - \log(1)} \right\}^{a} = 1 - \left\{ \frac{b}{b} \right\}^{a} = 1 - 1 = 0 \qquad (20)$$

Considering the lemma above and the proof following it, it is clear that the LOMINLIND has at least a mode and its pdf and cdf are valid functions.

3.2 Quantile function, median and simulation

According to Hyndman and Fan [30], the quantile function for any distribution is defined in the form $Q(u) = F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0 < u < 1$.

To derive the quantile function of the LOMINLIND, the cdf of the LOMINLIND is considered and inverted according to the above definition as follows:

$$F(x) = 1 - \beta^{\alpha} \left\{ \beta - \log\left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right\}^{-\alpha} = u \tag{21}$$

Simplifying equation (21) above gives:

$$-(\theta+1)\left(1 - e^{\beta - \beta(1-u)^{-\frac{1}{\alpha}}} \right) e^{-(\theta+1)} = -\frac{\theta + x + \theta x}{x} e^{-\frac{\theta + x + \theta x}{x}} \tag{22}$$

From the expression above, it can be seen that $-\dfrac{\theta + x + \theta x}{x}$ is the Lambert function of the real argument,

$-(\theta+1)\left(1 - e^{\beta - \beta(1-u)^{-\frac{1}{\alpha}}} \right) e^{-(\theta+1)}$ because the Lambert function is defined as: $w(x)e^{w(x)} = x$

Recall that the Lambert function has two branches with a branching point located at $\left(-e^{-1}, 1 \right)$. The lower branch, $W_{-1}(x)$ is defined in the interval $\left[-e^{-1}, 1 \right]$ and has a negative singularity for $x \to 0^{-1}$. The upper branch, $W_0(x)$, is defined for $x \in \left[-e^{-1}, \infty \right]$. Hence, equation (22) can be written as:

$$W\left(-(\theta+1)\left(1 - e^{\beta - \beta(1-u)^{-\frac{1}{\alpha}}} \right) e^{-(\theta+1)} \right) = -\frac{\theta + x + \theta x}{x} \tag{23}$$

Now for any $\theta > 0$ and $u \in (0,1)$, it follows that $\dfrac{\theta + x + \theta x}{x} > 1$ and $\left((\theta+1)\left(1 - e^{\beta - \beta(1-u)^{-\frac{1}{\alpha}}} \right) e^{-(\theta+1)} \right) < 0$. Therefore, considering the lower branch of the Lambert function, equation (50) can be presented as:

$$W_{-1}\left(-(\theta+1)\left(1 - e^{\beta - \beta(1-u)^{-\frac{1}{\alpha}}} \right) e^{-(\theta+1)} \right) = -\frac{\theta + x + \theta x}{x} \tag{24}$$

Collecting like terms in equation (24) and simplifying the result, the quantile function of the LOMINLIND is obtained as:

$$Q(u) = \left\{ -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(-(\theta+1)\left(1 - e^{\beta - \beta(1-u)^{-\frac{1}{\alpha}}}\right) e^{-(\theta+1)} \right) \right\}^{-1} \tag{25}$$

where u is a uniform variate on the unit interval *(0,1)* and $W_{-1}(.)$ represents the negative branch of the Lambert function.

Using (25) above, the median of *X* from the LOMINLIND is simply obtained by setting *u*=0.5 and this substitution of $u = 0.5$ in Equation (25) gives:

$$Median = \left\{ -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(-(\theta+1)\left(1 - e^{\beta - \beta(0.5)^{-\frac{1}{\alpha}}}\right) e^{-(\theta+1)} \right) \right\}^{-1} \tag{26}$$

Similarly, random numbers can be simulated from the LOMINLIND by setting $Q(u) = X$ and this process is called simulation using inverse transformation method. This means for any $\alpha, \beta, \theta > 0$ and $u \in (0,1)$:

$$X = \left\{ -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(-(\theta+1)\left(1 - e^{\beta - \beta(1-u)^{-\frac{1}{\alpha}}}\right) e^{-(\theta+1)} \right) \right\}^{-1} \tag{27}$$

"where u is a uniform variate on the unit interval *(0,1)* and $W_{-1}(.)$ represents the negative branch of the Lambert function".

Again using the function above, the quantile based measures of skewness and kurtosis are obtained as follows:

Kennedy and Keeping [31] defined the Bowley's measure of skewness based on quartiles as:

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{28}$$

And Moors [32] presented the Moors' kurtosis based on octiles by:

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{8}\right)} \tag{29}$$

"where $Q(.)$ is calculated by using the quantile function from equation (25).

3.3 Reliability analysis of the LOMINLIND

In this section, the survival (or reliability) function, the hazard (or failure) rate function, the cumulative hazard function, the reverse hazard function and the odds function are obtained for the LOMINLIND as follows:

The Survival function describes the probability that a unit, component or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{30}$$

Using the *cdf* of the LOMINLIND in (30) and simplifying the result, the survival function for the LOMINLIND is obtained as:

$$S(x) = \beta^\alpha \left\{ \beta - \log\left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right\}^{-\alpha} \tag{31}$$

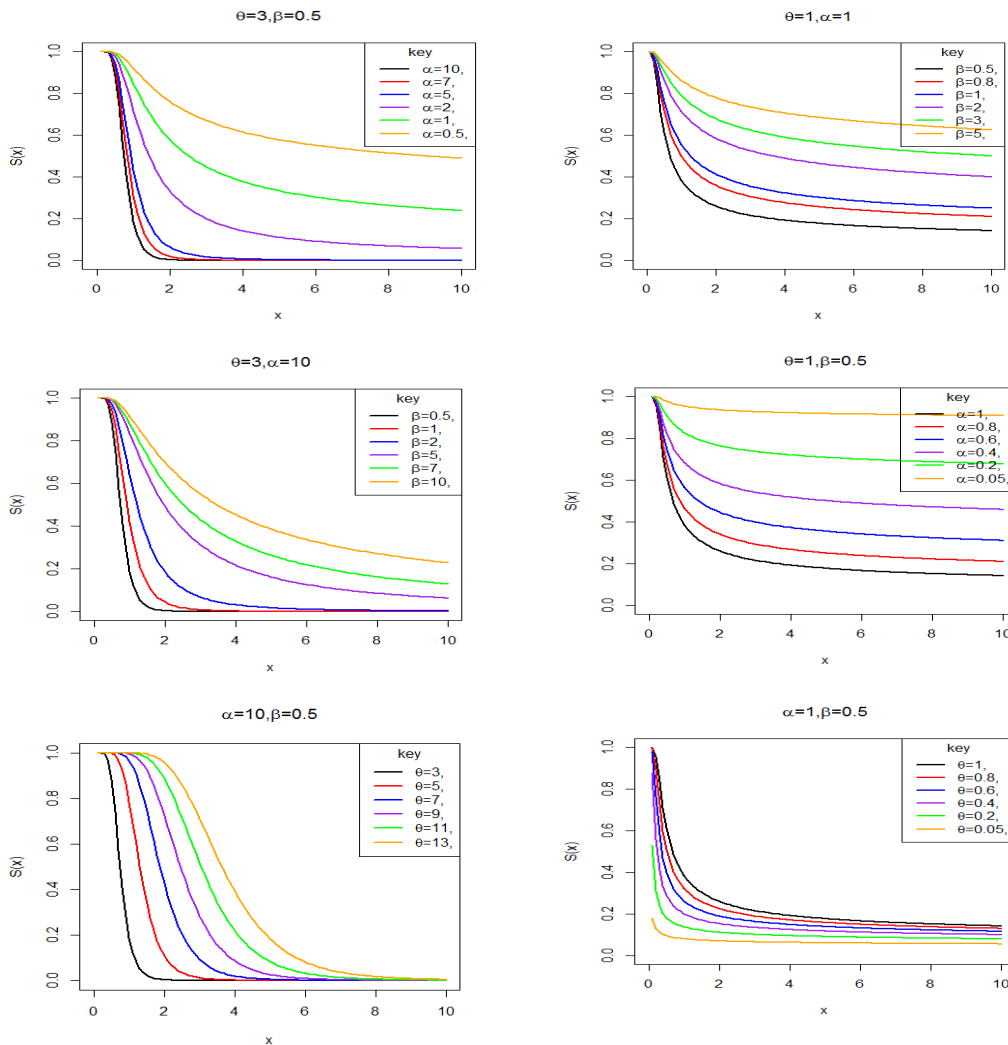The Fig. 3. is a plot for the survival function of the LOMINLIND using different parameter values.

Fig. 3. Survival function of LOMINLIND

Hazard function is also called failure rate function and it represents the likelihood that a component will fail for an interval of time. The hazard function is defined as:
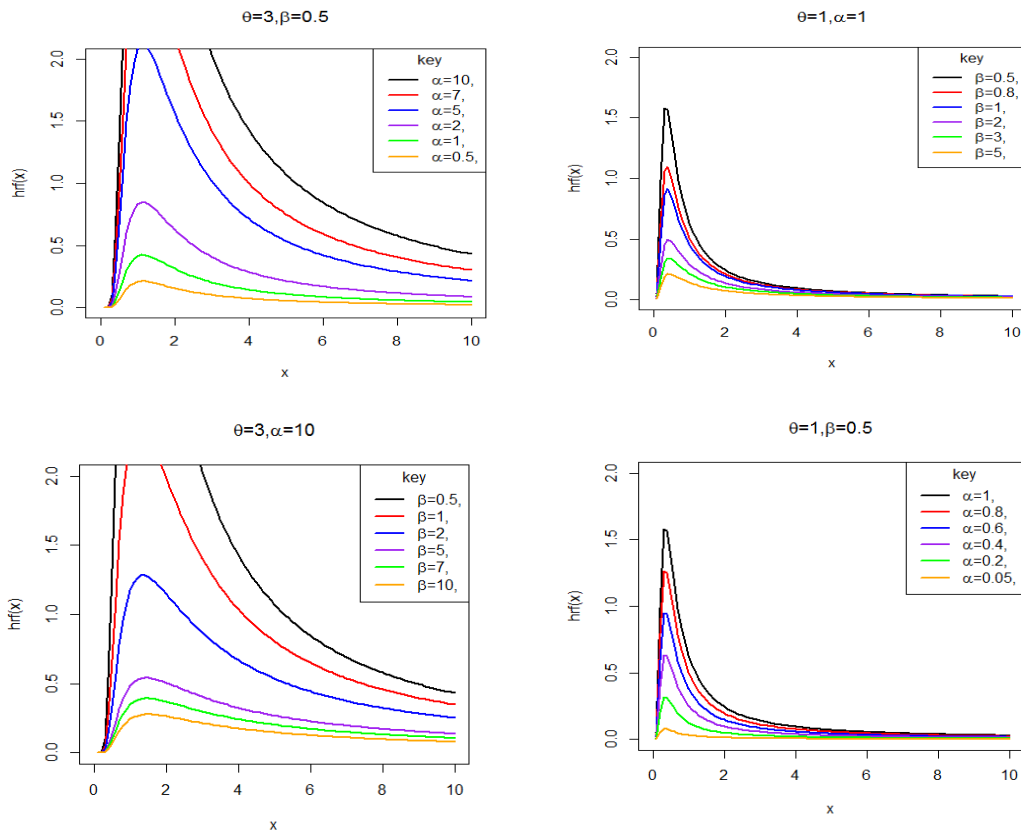
$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{32}$$

Making use of the pdf and cdf of LOMINLIND, an expression for the hazard rate of the LOMINLIND is simplified and given by:

$$h(x) = \frac{\alpha \dfrac{\theta^2}{\theta+1}\left(\dfrac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left(1-\left(\left(1+\dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right)^{-1}}{\left\{\beta - \log\left[1 - \left(\left(1 + \dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}} \tag{33}$$

where $x > 0, \alpha, \beta, \theta > 0$.

The following figure is a plot of the hazard function for some arbitrary parameter values.
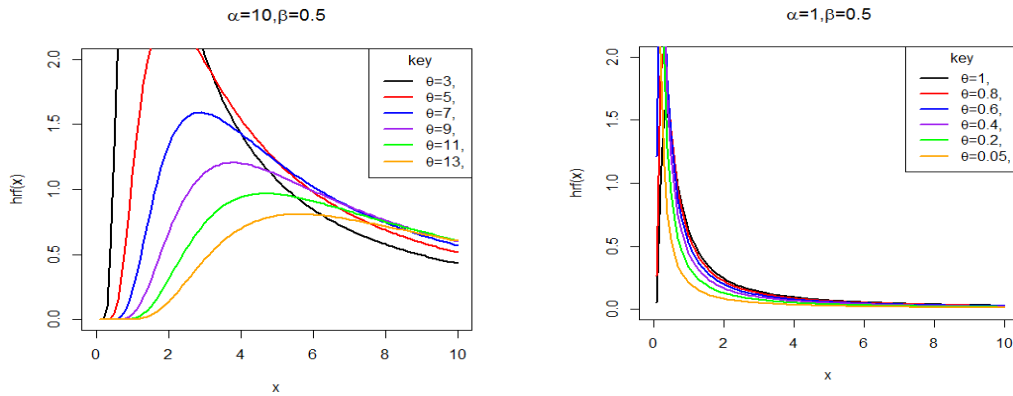
Fig. 4. Hazard function of LOMINLIND

The cumulative hazard function of a variable or unit is a function that generates a cumulative hazard value which corresponds to the sum of all the hazard values for failed units with ranks up to and including that failed unit. The cumulative hazard function is defined as:

$$H(x) = \int_0^x h(t)\,dt = \int_0^x \frac{f(t)}{1-F(t)}\,dt = -\ln S(x) \tag{34}$$

Substituting the cdf of the LOMINLIND in (34), the cumulative hazard function for the LOMINLIND is obtained as:

$$H(x) = -\ln\left\{ \beta^\alpha \left\{ \beta - \log\left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right\}^{-\alpha} \right\} \tag{35}$$

The reversed hazard rate (Rh(x)) is defined as the ratio of the density function to the distribution function of a random variable. The reversed hazard function of a variable is mathematically defined as:

$$Rh(x) = \frac{f(x)}{F(x)} \tag{36}$$

Again, substituting the pdf and cdf of the LOMINLIND in (36), the reverse hazard function of the LOMINLIND is expressed as:

$$Rh(x) = \frac{\dfrac{\theta^2}{\theta+1}\left(\dfrac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left\{\beta - \log\left[1 - \left(\left(1 + \dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}^{-1}}{\left[1 - \left(\left(1 + \dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\left\{\left(\beta - \log\left[1 - \left(\left(1 + \dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right)^\alpha - \beta^\alpha\right\}} \tag{37}$$

The odds function of a random variable X is a measure of the ratio of the probability that the variable or unit will survive beyond x to the probability that it will fail before x. It is obtained by dividing the cdf by the reliability (survival) function. That is:

$$O(x) = \frac{F(x)}{1-F(x)} = \frac{F(x)}{S(x)}$$

(38)

Using the cdf of the LOMINLIND in (38), the odds function for the distribution is given as:

$$O(x) = \beta^{-\alpha} \left\{ \left(\beta - \log\left[1 - \left(\left(1 + \frac{\theta}{(\theta+1)x} \right) e^{-\frac{\theta}{x}} \right) \right] \right)^{\alpha} - \beta^{\alpha} \right\}$$

(39)

where $x > 0, \alpha, \beta, \theta > 0$.

3.4 Order statistics

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the LOMINLIND and let $X_{1:n}, X_{2:n}, \ldots, X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The *pdf*, $f_{i:n}(x)$ of the ith order statistic can be obtained by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} f(x)F(x)^{k+i-1}$$

(40)

Using (6) and (7), the *pdf* of the ith order statistics $X_{i:n}$, can be expressed from (40) as:

(41)

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left[\frac{\alpha\beta^\alpha \frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left(1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right)^{-1}}{\left\{\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}^{\alpha+1}} \right] \left[1 - \left\{ \frac{\beta}{\left(\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right)} \right\}^{\alpha} \right]^{i+k-1}$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the LOMINLIND are respectively given by:

$$f_{1:n}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[\frac{\alpha\beta^\alpha \frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{\frac{\theta}{x}}\left(1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{\frac{\theta}{x}}\right)\right)^{-1}}{\left\{\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{\frac{\theta}{x}}\right)\right]\right\}^{\alpha+1}} \right] \left[1 - \left\{ \frac{\beta}{\left(\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{\frac{\theta}{x}}\right)\right]\right)} \right\}^{\alpha} \right]^{k}$$

(42)

and

$$f_{n:n}(x) = n \left[\frac{\alpha\beta^\alpha \frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left(1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right)^{-1}}{\left\{\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}^{\alpha+1}} \right] \left[1 - \left\{ \frac{\beta}{\left(\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right)} \right\}^{\alpha} \right]^{n-1}$$

(43)

4 Maximum Likelihood Estimation of the Unknown Parameters of the LOMINLIND

Let $X_1, X_2, \ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the LOMINLIND with unknown parameters, α, β and θ defined previously.

The likelihood function of the LOMINLIND using the pdf in equation (7) is given by:

$$L\left(\underline{X}/\alpha,\beta,\theta\right)=\left(\frac{\alpha\beta^{\alpha}\theta^2}{\theta+1}\right)^n\prod_{i=1}^{n}\left(\frac{\left(\frac{1+x_i}{x_i^3}\right)e^{-\frac{\theta}{x_i}}\left(1-\left(\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\right)^{-1}}{\left\{\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\right]\right\}^{\alpha+1}}\right) \tag{44}$$

Let the natural logarithm of the likelihood function be, $l(\kappa)=\log L\left(\underline{X}\mid\alpha,\beta,\theta\right)$, therefore, taking the natural logarithm of the function above gives:

$$l(\kappa)=n\log\alpha+\alpha n\log\beta+2n\log\theta-n\log(\theta+1)+\sum_{i=1}^{n}\log\left(\frac{1+x_i}{x_i^3}\right)-\theta\sum_{i=1}^{n}\frac{1}{x_i}-\sum_{i=1}^{n}\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\right]$$

$$-(\alpha+1)\sum_{i=1}^{n}\log\left\{\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\right]\right\} \tag{45}$$

Differentiating $l(\kappa)$ partially with respect to α, β and θ respectively gives the following results:

$$\frac{\partial l(\kappa)}{\partial\alpha}=\frac{n}{\alpha}+n\log\beta-\sum_{i=1}^{n}\log\left\{\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\right]\right\} \tag{46}$$

$$\frac{\partial l(\kappa)}{\partial\beta}=\frac{\alpha n}{\beta}-(\alpha+1)\sum_{i=1}^{n}\left\{\left(\beta-\log\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\right]\right)^{-1}\right\} \tag{47}$$

$$\frac{\partial l(\kappa)}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{(\theta+1)}-\sum_{i=1}^{n}\frac{1}{x_i}-\sum_{i=1}^{n}\left[\frac{\left(\frac{1}{x}+\frac{\theta}{(\theta+1)^2x_i}\left(\frac{\theta+1-x_i}{x_i}\right)\right)e^{\frac{\theta}{x_i}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{\frac{\theta}{x_i}}\right]}\right]+(\alpha+1)\sum_{i=1}^{n}\left[\frac{\left(\frac{1}{x}+\frac{\theta}{(\theta+1)^2x_i}\left(\frac{\theta+1-x_i}{x_i}\right)\right)e^{\frac{\theta}{x_i}}}{\left[\beta-\log\left[1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{\frac{\theta}{x_i}}\right]\right]\left[1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{\frac{\theta}{x_i}}\right]}\right] \tag{48}$$

Making equation (46), (47) and (48) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters α, β and θ. However, these solutions cannot be obtained manually except numerically with the aid of suitable statistical software like R, SAS, MATHEMATICA e.t.c.

5 Applications to Lifetime Datasets

In this section, we present three real life datasets, their summary and applications. The section also fits the proposed distribution (LOMINLIND) together with other three models which include Inverse Lindley distribution (INLIND), Lindley distribution (LIND) and the Lomax distribution (LOMD) to the three lifetime datasets. The density functions of these distributions are given as:

5.1 Lomax Inverse Lindley Distribution (LOMINLIND)

The *pdf* of the LOMINLIND distribution is given as:

$$f(x) = \frac{\alpha\beta^{\alpha} \dfrac{\theta^2}{\theta+1}\left(\dfrac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left(1-\left(\left(1+\dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right)^{-1}}{\left\{\beta-\log\left[1-\left(\left(1+\dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\right]\right\}^{\alpha+1}} \tag{49}$$

5.2 The Inverse Lindley Distribution (INLIND)

The *pdf* of the INLINDD is given as:

$$f(x) = \frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}} \tag{50}$$

5.3 The Lindley Distribution (LIND)

The *pdf* of the LIND is given as:

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \tag{51}$$

5.4 The Lomax Distribution (LOMD)

The *pdf* of the LOMD is given as:

$$f(x) = \frac{\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \tag{52}$$

To evaluate and compare the above stated distributions, some model selection criteria which include the value of the log-likelihood function evaluated at the maximum likelihood estimates (ℓ), Akaike Information Criterion, *AIC,* Consistent Akaike Information Criterion, *CAIC*, Bayesian Information Criterion, *BIC*, Hannan Quin Information Criterion, *HQIC*, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics are being considered. The details about the statistics A*, W* and K-S are discussed in Chen and Balakrishnan [33]. Some of these statistics are computed with the following formulas:

$$AIC = -2\ell + 2k \text{ , } BIC = -2\ell + k\log(n), CAIC = -2\ell + \frac{2kn}{(n-k-1)} \text{ and } HQIC = -2\ell + 2k\log\left[\log(n)\right]$$

Where ℓ denotes the value of log-likelihood function evaluated at the *MLEs*, k is the number of model parameters and n is the sample size. Note, when taking our decisions it is considered that any model with the lowest values of these statistics to be the best model that fit the dataset. The required computations are carried out using the R package "AdequacyModel" from R Core Team [34] which is freely available from http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.

Table 2 (for dataset I), 6 (for dataset II) and 10 (for dataset III) list the Maximum Likelihood Estimates of the model parameters whereas the statistics AIC, CAIC, BIC, HQIC, A*, W* and K-S for the fitted LOMINLIND, INLIND, LIND and LOMD models are given in Tables 3 & 4 for dataset I, 7 & 8 for dataset II and 11&12 for dataset III respectively.

Data set I: This data set represents the relief times (in minutes) of 20 patients receiving an analgesic reported by Gross et al. [35] and has been used by shanker et al. [36] and Ieren et al., [37]. Its values are given as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0. The summary of the data set is provided in Table 1.

Table 1. Summary statistics for the dataset I

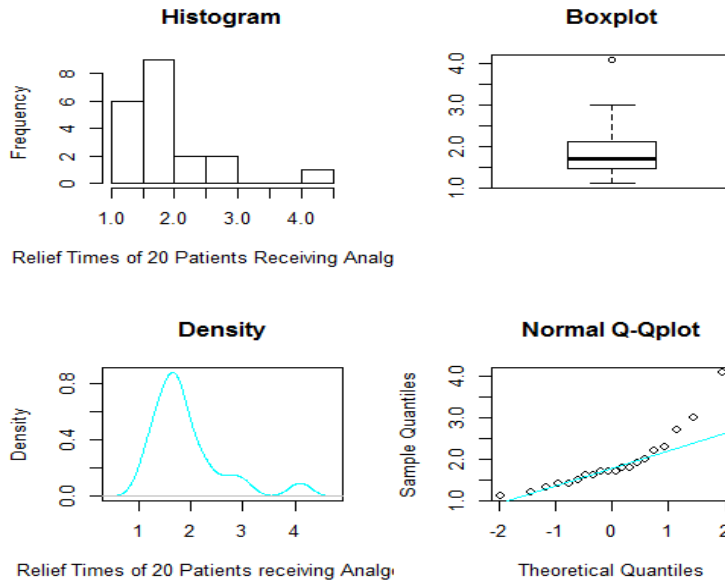| N	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
20	1.10	1.475	1.70	2.05	1.90	4.10	0.4958	1.8625	7.1854

Fig. 5. A graphical summary of dataset I

Considering the summary statistics in Table 1 and the histogram, box plot, density and normal Q-Q plot in Fig. 5, it is clear that dataset I is positively skewed.

Dataset II: This data set represents the strength of 1.5cm glass fibers initially collected by members of staff at the UK national laboratory. It has been used by Afify and Aryal [38], Barreto-Souza et al. [39],

Bourguignon et al. [40], Oguntunde et al. [41], Ieren and Yahaya [42] as well as Smith and Naylor [43]. Its summary is given as follows:

Table 2. Maximum likelihood parameter estimates for dataset I

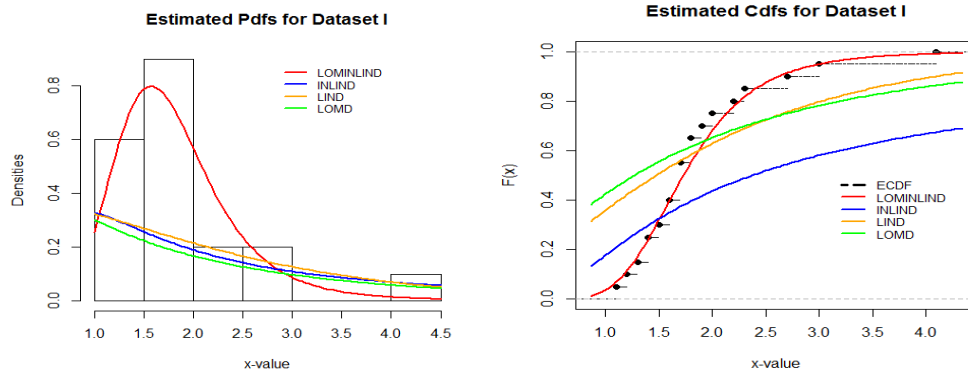| Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
LOMINLIND	7.6010832	4.2489000	0.1061194
INLIND	2.254666	-	-
LIND	0.8158912	-	-
LOMD	-	5.636623	9.689862

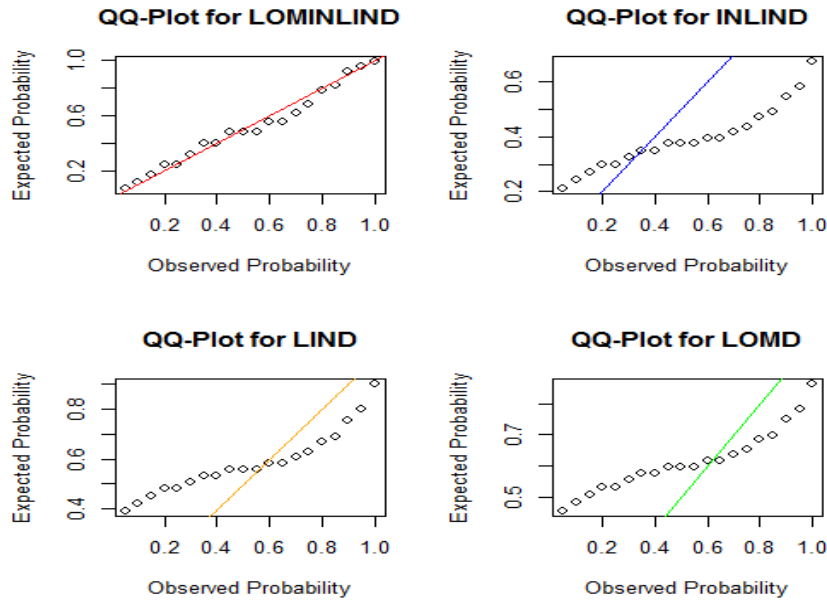**Fig. 6. Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset I**

Fig. 7. Probability plots for the fit of the LOMINLIND, INLIND, LIND & LOMD based on dataset I

Table 3. The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset I

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
LOMINLIND	16.39677	38.79354	40.29354	41.78073	39.37667	1st
INLIND	31.75719	65.51438	65.73661	66.51012	65.70876	3rd
LIND	30.24955	62.4991	62.72132	63.49483	62.69348	2nd
LOMD	34.38269	72.76537	73.47126	74.75684	73.15413	4th

Table 4. The A^*, W^*, K-S statistic and P-values for dataset I

Distribution	A^*	W^*	K-S	P-Value (K-S)	Ranks
LOMINLIND	0.289549	0.04939996	0.099046	0.9895	1st
INLIND	0.2711856	0.04666718	0.36946	0.008507	2nd
LIND	0.6757989	0.1140868	0.39096	0.004424	3rd
LOMD	0.5486187	0.09268843	0.45452	0.0005155	4th

Table 5. Summary statistics for dataset II

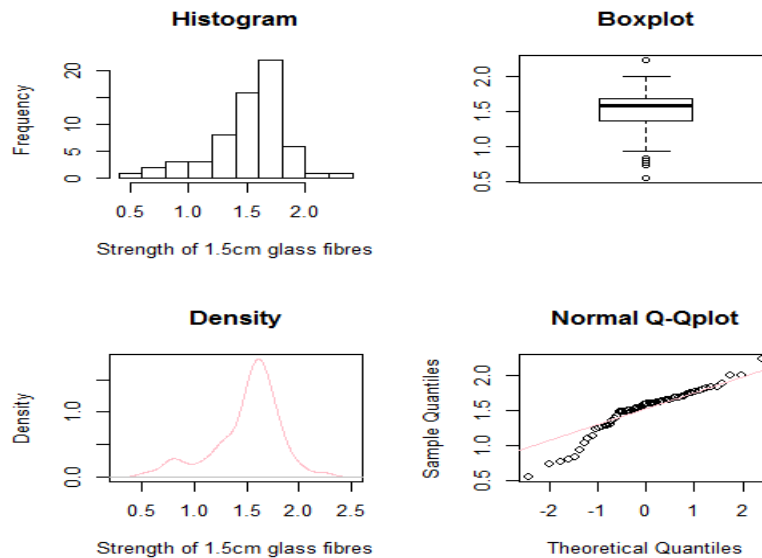| n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
63	0.550	1.375	1.590	1.685	1.507	2.240	0.105	-0.8786	3.9238

Fig. 8. A graphical summary of dataset II

Based on the summary statistics in Table 5 and the plots in Fig. 8 it can be seen that the second dataset (dataset II) is negatively skewed or skewed to the left.

Dataset III: Actuarial Science (Mortality Deaths) data.

This third dataset represents 280 observations on the age of death (in years) of retired women with temporary disabilities. These dataset has been studied by Balakrishnan et al. [44]. It is important for the Mexican Institute of Social Security (IMSS) to study the distributional behavior of the mortality of retired

people on disability because it enables the calculation of long and short term financial estimation, such as the assessment of the reserve required to pay the minimum pensions.

The data corresponding to lifetimes (in years) of retired women with temporary disabilities who died during 2004 and which are incorporated in the Mexican insurance public system are: 22, 24, 25(2), 27, 28, 29(4), 30, 31(6), 32(7), 33(3), 34(6), 35(4), 36(11), 37(5), 38(3), 39(6), 40(14),41(12), 42(6), 43(5), 44(7), 45(10), 46(6), 47(5), 48(11), 49(8), 50(8), 51(8), 52(14), 53(10), 54(13), 55(11), 56(10), 57(15), 58(11), 59(9), 60(7), 61(2), 62, 63, 64(4), 65(2), 66(3), 71, 74, 75, 79, 86. Its summary is given as follows:

Table 6. Maximum likelihood parameter estimates for dataset II

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
LOMINLIND	8.53537822	7.91781787	0.06757877
INLIND	1.897071	-	-
LIND	0.9964413	-	-
LOMD	-	7.109656	9.882660

Table 7. The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset II

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
LOMINLIND	25.02074	56.04148	56.44148	62.51813	58.59296	1st
INLIND	87.28032	176.5606	176.6252	178.7195	177.4111	3rd
LIND	82.58534	167.1707	167.2352	169.3296	168.0212	2nd
LOMD	94.55355	193.1071	193.3038	197.4249	194.8081	4th

Table 8. The A*, W*, K-S statistic and P-values for dataset II

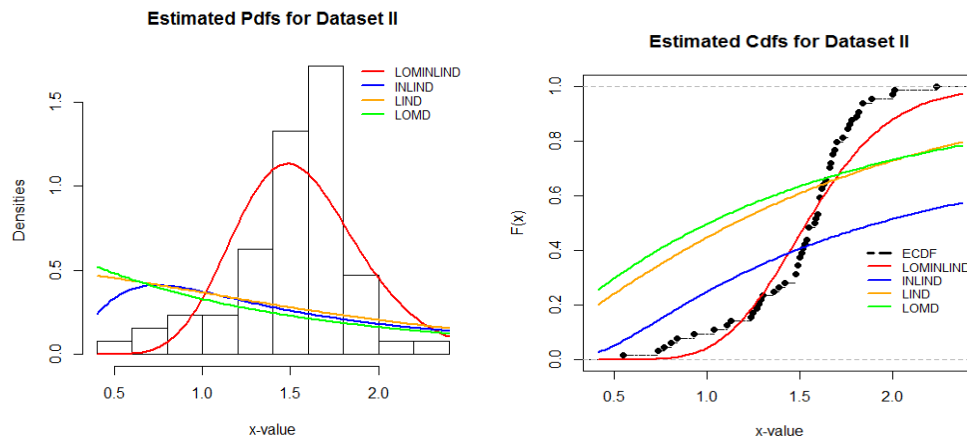| Distribution	A*	W*	K-S	P-Value (K-S)	Ranks
LOMINLIND	3.255563	0.5960105	0.15256	0.1017	1st
INLIND	4.94591	0.9156447	0.46846	1.264e-12	3rd
LIND	3.078766	0.562949	0.38882	7.887e-09	2nd
LOMD	3.393551	0.6209432	0.42783	1.337e-10	4th

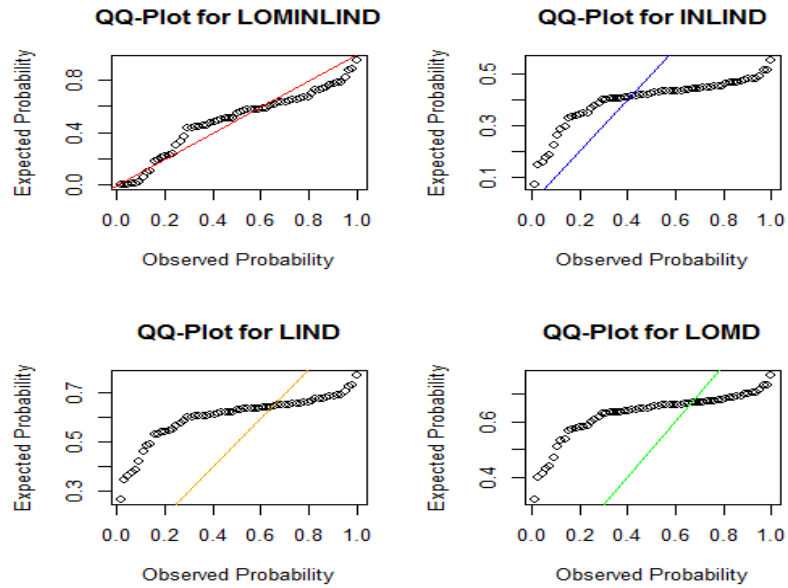**Fig. 9. Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset II**

Fig. 10. Probability plots for the fit of the LOMINLIND, INLIND, LIND & LOMD based on dataset II

Table 9. Descriptive statistics for dataset III

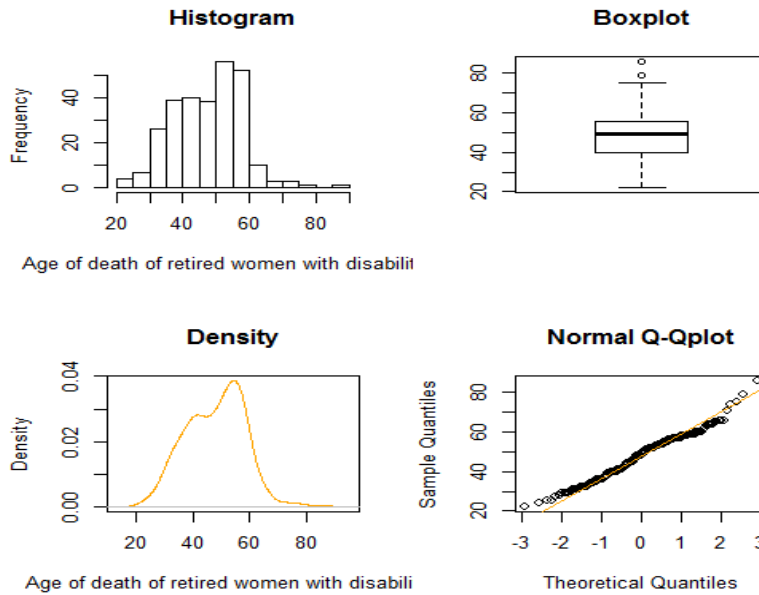| n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
280	22.00	40.00	49.00	55.25	47.79	86.00	108.63	0.06703	0.0524

Fig. 11. A graphical summary of dataset III

Again, the descriptive statistics in Table 9 and the histogram, box plot, density and normal Q-Q plot in Fig. 11 reveal that the third dataset (dataset III) is approximately normal and is not a skewed dataset.

Looking at the tabulated values of the different model selection criteria, AIC, CAIC, BIC, HQIC, A*, W* and K-S given in Tables 3 and 4 as well as 7 and 8 for dataset I and II respectively, one can clearly see that the LOMINLIND has the least AIC, CAIC, BIC, HQIC, A*, W* and K-S values for dataset I and II which are positively and negatively skewed respectively. Also, the histogram for the datasets with fitted probability density functions and estimated cumulative distribution functions shown in Figs. 6 and 9 for dataset I and II respectively as well as the Q-Q plots in Figs. 7 and 10 for dataset I and II respectively confirm that the LOMINLIND fits the two datasets better than the INLIND, LIND and LOMD irrespective of the difference in the nature of the datasets which shows that it is more flexible compared to the other three distributions (INLIND, LIND and LOMD).

Table 10. Maximum likelihood parameter estimates for dataset III

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
LOMINLIND	9.331296	4.784396	7.968393
INLIND	7.051431	-	-
LIND	0.0410294	-	-
LOMD	-	0.5294455	9.1751334

Table 11. The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset III

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
LOMINLIND	1574.222	3154.444	3154.531	3165.348	3158.818	3rd
INLIND	1678.924	3359.847	3359.862	3363.482	3361.305	4th
LIND	1266.843	2535.686	2535.7	2539.321	2537.144	1st
LOMD	1573.152	3150.304	3150.347	3157.574	3153.22	2nd

Table 12. The A*, W*, K-S statistic and P-values for dataset III

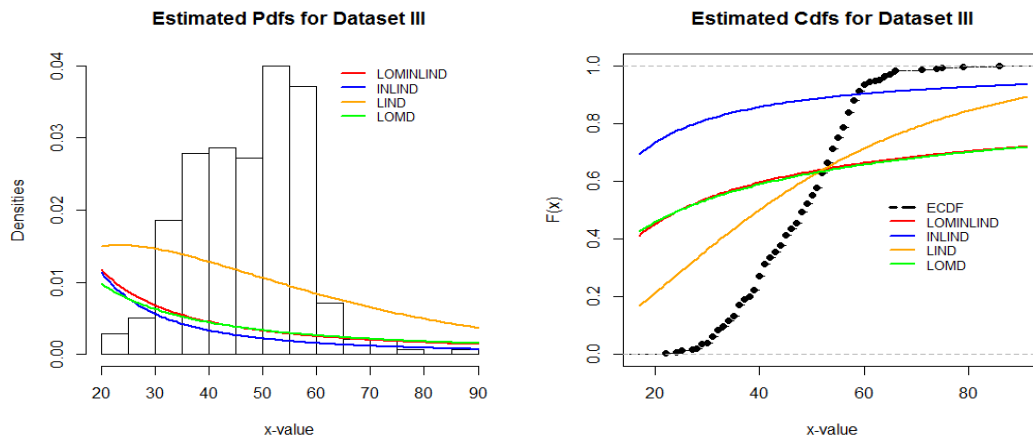| Distribution	A*	W*	K-S	P-Value (K-S)	Ranks
LOMINLIND	4.515244	0.7927554	0.51192	2.2e-16	3rd
INLIND	4.190734	0.7375098	0.7864	2.2e-16	4th
LIND	2.654785	0.4713172	0.33796	2.2e-16	1st
LOMD	3.969601	0.6999846	0.50848	2.2e-16	2nd

Fig. 12. Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset III
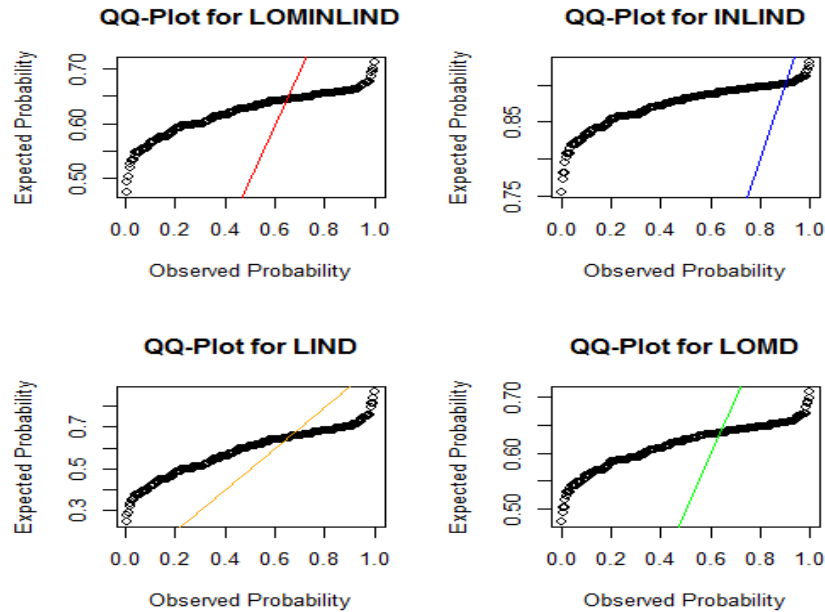
23

Fig. 13. Probability plots for the fit of the LOMINLIND, INLIND, LIND & LOMD based on dataset III

Based on the results for the two datasets above, it is true that the Lomax generator of distributions by Cordeiro et al., [25] has the capacity to increase the flexibility of continuous probability models just as previously reported by other authors such as Venegas et al., [28], Omale et al. [27], Ieren et al., [29] as well as Ieren and Kuhe [26].

Now, in Tables 11 and 12 for dataset III, the result is the opposite of what is obtained in dataset I and dataset II and this is because the third dataset (dataset III) is approximately normal. Similarly, the histogram for the dataset (dataset III) with fitted probability density functions and estimated cumulative distribution functions shown in Fig. 12 as well as the Q-Q plots in Fig. 13 confirm that the LOMINLIND is not a good model for the third data (dataset III) because it is not a skewed data and this also prove the fact that the Lomax generator of distributions by Cordeiro et al., [25] produces distributions with high degree of skewness compared to their conventional counterparts.

6 Summary and Conclusion

This article proposed Lomax-Inverse Lindley distribution (LOMINLIND) which becomes an extension of the Inverse Lindley distribution with a study of its properties such as Limiting behavior, quantile function for calculation of median and simulation, survival function, hazard function with related features and expressions for the distribution of minimum and maximum order statistics. The parameters of the proposed distribution have been estimated using the method of maximum likelihood estimation. The graphs of the pdf of the distribution show that it is skewed and that its shape depends on the values of the parameters. Also, the plots of the survival function implies that the function is decreasing, that is, the distribution could be used to model age-dependent or time-dependent events or variables (where probability of survival decreases as time advances or where survival rate decreases with increase in age or time). Also, the hazard rate of the distribution is always unimodal and this repeated unimodal shape shows that the new distribution would be a flexible model for survival analysis of events whose failure rate increases exponentially at first and then slowly decreases with time in the end. Three lifetime datasets were used to check the usefulness and

applicability of the new distribution and based on the results of the applications, it was found that the Lomax-Inverse Lindley distribution is more appropriate for skewed datasets compared to the Inverse Lindley, Lindley and Lomax distributions and could be used as an alternative in real life situations.

Competing Interests

Authors have declared that no competing interests exist.

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