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A Modified Solution of the Nonlinear Singular Oscillator by Extended Iteration Procedure

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Authors' contributions

This work was carried out in collaboration between the authors. Author BMIH designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author MMAH managed the analyses of the study and managed the literature searches. Both of the authors read and approved the final manuscript.

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Abstract

A modified solution of the nonlinear singular oscillator has been obtained based on the extended iteration procedure. We have used an appropriate truncation of the obtained Fourier series in each step of iterations to determine the approximate analytic solution of the oscillator. The third approximate frequency of the nonlinear singular oscillator shows a good agreement with its exact values. Earlier different authors presented the analytic solution of the oscillator by using various types of methods. We have compared the results obtained by the modified technique with some of the existing results. We see that some of their techniques deviate from higher-order approximations and the present technique performs comparatively better. The rate of change of percentage of error of the presented modified solution shows the validity of convergence.

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1 Introduction

The subject of differential equations is both one of the most beautiful parts of mathematics and essential tool for modelling many physical situations like mechanical vibration, nonlinear circuits, chemical oscillation, and space dynamics and so on. These equations have shown their usefulness in the field of ecology, business cycle and biology. However, to solve the corresponding differential equations, the solution of such problems remains necessary. The differential equations may be linear or non-linear, autonomous or non-autonomous. Pragmatically, a lot of differentiation that includes physical phenomena are nonlinear. The solutions procedures of linear differential equations are comparatively easy and highly developed. On the contrary, it is very little known of a general character about nonlinear equations. Ordinarily, the nonlinear problems are solved by converting it into linear equations attributing some terms; but such linearization is not possible or feasible at all times. In these circumstances, there are a few analytical approaches to find approximate solutions to nonlinear problems, for instance; perturbation [1-5], standard as well as modified Linstedt-Poincare [6], Harmonic balance [7-11], Homotopy [12], Iterative [13-27] methods, He’s new amplitude-frequency relationship [28], Jacobi collocation [29] etc. Among them, the perturbation method is the most widely utilized method in which the nonlinear term is small. Recently, some authors utilize an iteration procedure [13-27] which is valid for small together with a large amplitude of oscillation, to attain the approximate frequency and the harmonious periodic solution of such nonlinear problems. A few numbers of scientists used a modified version of this process to develop the results; luckily the extended iteration method sometimes improves the results when the functions are not differentiable which is for the singular oscillator. The method of iteration is developed by R.E. Mickens, Lim, Hu and Wu.

The main purpose of this paper is to ease the simplification process and improve the accuracy of the approximate analytic solution of ‘Nonlinear Singular Oscillator’ by extended iteration procedure so that it can help us to investigate the nature (amplitude, frequency *etc*) in the nonlinear dynamical systems.

2 The Method

Let us suppose the general form of a nonlinear oscillator modelled by

$$\ddot{x} + f(x) = 0 \quad x(0) = A \quad \dot{x}(0) = 0 \tag{1}$$

where over dots denote differentiation concerning time t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ on both sides of Eq. (1), we obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(x) \equiv G(x). \tag{2}$$

Now, formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k) + (x_k - x_{k-1})\, G_x(x_k) \qquad k = 0, 1, 2, \ldots \tag{3}$$

where $$G_x = \frac{\partial G}{\partial x} \text{ and } x_0(t) = A\cos(\Omega_0 t) = A\cos\theta \tag{4}$$

and x_{k+1} satisfies the conditions

$$x_{k+1}(0)=A, \quad \dot{x}_{k+1}(0)=0. \tag{5}$$

At each stage of the extended iteration, Ω_k is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$. The above procedure gives the sequence of solutions which are mentioned here $x_0(t),\ x_1(t),\ \cdots$ by. The method can proceed to any order of approximation, but due to growing algebraic complexity, the solution is confined to a lower order usually to the second [13].

3 Solution Procedure

Let us consider the nonlinear singular Oscillator

$$\ddot{x}+x^{-1}=0. \tag{6}$$

Adding $\Omega^2 x$ on both sides of Eq. (6), we get

$$\ddot{x}+\ \Omega^2 \mathrm{x}=\Omega^2 \mathrm{x}\text{-}\ x^{-1}=G(x) \tag{7}$$

where $G(x)=\Omega^2 \mathrm{x}\text{-}\ x^{-1}$.

Therefore $G_x=\Omega^2+x^{-2}$.

According to Eq. (3), the extended iteration scheme of Eq. (7) is

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=(\Omega_k^2 x_k-x_k^{-1})+(x_k-x_0)(\Omega_k{}^2+x_0{}^{-2}) \tag{8}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained by putting k=0 in Eq.(8) and using Eq.(4) we get

$$\ddot{x}_1+\Omega_0^2 x_1=\Omega_0^2 A\cos\theta-(A\cos\theta)^{-1} \tag{9}$$

Now expanding $(\cos\theta)^{-1}$ in a truncated Fourier cosine series, Eq. (9) reduces to

$$\ddot{x}_1+\Omega_0^2 x_1=(\Omega_0^2 A-\frac{2}{A})\cos\theta+\frac{2}{A}\cos 3\theta. \tag{10}$$

Now secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

$$\text{i.e.}\ \Omega_0=\frac{\sqrt{2}}{A}=\frac{1.41421}{A}. \tag{11}$$

This is the first approximate frequency of the oscillator.

After simplification the Eq. (10) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \frac{2}{A}\cos 3\theta \,. \tag{12}$$

The complete solution is

$$x_1(t) = C\cos\theta - \frac{A}{8}\cos 3\theta \,. \tag{13}$$

Using $x_1(0) = A$, we have $C = \frac{9}{8}A$.

Therefore $x_1(t) = A(\frac{9}{8}\cos\theta - \frac{1}{8}\cos 3\theta)$. (14)

This is the first approximate solution of the oscillator.

Proceeding to the second level of iteration $x_2(t)$ satisfies the equation

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 &= (\Omega_1^2 x_0 - x_0^{-1}) + (\Omega_1^{\,2} + x_0^{-2})(x_1 - x_0) \\ &= \Omega_1^{\,2} x_1 + x_1 x_0^{-2} - 2x_0^{-1}\end{aligned}\,. \tag{15}$$

Now expanding second and third term on the right-hand side in a truncated Fourier cosine series, Eq. (15) reduces to

$$\ddot{x}_2 + \Omega_1^2 x_2 = (\Omega_1^{\,2} A\alpha + \frac{2\alpha - 4 - 2\beta}{A})\cos\theta + (\Omega_1^{\,2} A\beta - \frac{2\alpha - 4 - 6\beta}{A})\cos 3\theta \,, \tag{16}$$

where $\alpha = \frac{9}{8}, \beta = -\frac{1}{8}$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

i.e. $$\Omega_1 = \frac{1.1547}{A} \tag{17}$$

This is the second approximate frequency of the oscillator.

After simplification Eq. (16) reduces to

$$\ddot{x}_2 + \Omega_1^2 x_2 = (\Omega_1^{\,2} A\beta - \frac{2\alpha - 4 - 6\beta}{A})\cos 3\theta \tag{18}$$

$$\ddot{x}_2 + \Omega_1^2 x_2 = \frac{0.833}{A}\cos 3\theta \tag{19}$$

The complete solution of Eq. (16) is

$$x_2(t) = C\cos\theta - 0.078125A\cos 3\theta \tag{20}$$

Using $x_2(0) = A$, we have $C = 1.078125A$

Therefore,

$$\begin{aligned} x_2(t) &= 1.078125A\cos\theta - 0.078125A\cos 3\theta \\ &= \alpha_1 A\cos\theta + \beta_1 A\cos 3\theta, \end{aligned} \tag{21}$$

where $\alpha_1 = 1.078125$ and $\beta_1 = -0.078125$

Proceeding to the third level of iteration $x_2(t)$ satisfies the equation

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2{}^2 x_2 + x_2 x_0{}^{-2} - 2x_0{}^{-1} \tag{22}$$

Now expanding the term on the right-hand side in a Fourier cosine series, the Eq. (22) reduces to

$$\ddot{x}_3 + \Omega_2^2 x_3 = (\Omega_2{}^2 A\alpha_1 + \frac{2\alpha_1 - 4 - 2\beta_1}{A})\cos\theta + (\Omega_2{}^2 A\beta_1 - \frac{2\alpha_1 - 4 - 6\beta_1}{A})\cos 3\theta \tag{23}$$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

$$\Omega_2 = \frac{1.2511}{A} \tag{24}$$

4 Results and Discussion

An extended iterative method is presented to obtain the approximate solution of the nonlinear singular oscillator. In this section, we have expressed the accuracy of the extended iteration method by comparing with the existing results from different methods and with the exact frequency of the singular oscillator. To demonstrate the accurateness, we have calculated the percentage of errors (denoted by Er (%)) by the definitions.

$$\text{Error} = \left|\frac{\Omega_e - \Omega_k}{\Omega_e}\right| \times 100\%$$

where $\Omega_k;\ k = 0,1,2,\cdots,$ represents the approximate frequencies obtained by the present method and Ω_e represents the corresponding exact frequency of the oscillator.

Here we have calculated the first, second and third approximate frequencies Ω_0, Ω_1 and Ω_2 respectively.

To compare the approximate frequencies, we have also given the existing results determined by Mickens iteration method [18], Mickens HB method [9] and Haque's *et al* iteration method [19], shown in the following Table 1.

We see that the percentage error of the second approximate frequency is greater than the percentage error of the first approximate frequency obtained by Mickens [18]. Also, the percentage error of the second approximate frequencies obtained by Mickens harmonic balance method [9] and Haque's *et al* iteration method [19] are greater than the percentage error of second approximate frequencies obtained by the adopted iteration method. Therefore, the current method gives a significantly better result than other methods.

Table 1. Comparison of the approximate frequencies with exact frequency

Comparison of the approximate frequencies with exact frequency Ω_e [18] of $\ddot{x}+x^{-1}=0$			
	Exact frequency $\Omega_{exact}=\Omega_e=\frac{1.253}{A}$		
Amplitude *A*	**First approximate frequencies & Error (%)**	**Second approximate frequencies & Error (%)**	**Third approximate frequencies & Error (%)**
Mickens iteration method [18]	$\Omega_{MI0}=\frac{1.155}{A}$ 7.9	$\Omega_{MI1}=\frac{1.018}{A}$ 18.1	-----
Mickens HB method [9]	$\Omega_{MH0}=\frac{1.414}{A}$ 12.84	$\Omega_{MH1}=\frac{1.273}{A}$ 1.6	$\Omega_{MH2}=\frac{1.273}{A}$ 1.58
Haque's *et al* iteration method [19]	$\Omega_{HI0}=\frac{1.414}{A}$ 12.84	$\Omega_{HI1}=\frac{1.208}{A}$ 3.63	$\Omega_{HI2}=\frac{1.265}{A}$ 0.92
Adopted method	$\Omega_0=\frac{1.414}{A}$ 12.84	$\Omega_1=\frac{1.155}{A}$ 7.85	$\Omega_2=\frac{1.251}{A}$ 0.15

$\Omega_0, \Omega_1, \Omega_2$ respectively denote the first, second and third modified approximate frequencies; Ω_{MI0} and Ω_{MI1} denote the first and second frequencies obtained by Mickens iteration method [18]; Ω_{MH0}, Ω_{MH1} and Ω_{MH2} denote the first, second and third frequencies obtained by Mickens HB method [9]; Ω_{HI0}, Ω_{HI1} and Ω_{HI2} denote the first, second and third frequencies obtained by Haque's iteration method [19]. Error (%) denotes percentage error.

5 Convergence and Consistency Analysis

In this article, the result has been improved only by rearranging the governing equation. Although in most of the earlier articles, the results have been improved by modifying the method. So the not only modification of the model is important but also rearranging is important in the case of the iteration procedure.

We know that the basic idea of Iterative methods is to construct a sequence of solutions x_k (as well as frequencies Ω_k) that have the property of convergence

$$x_e = \lim_{k \to \infty} x_k \qquad \text{or,} \quad \Omega_e = \lim_{k \to \infty} \Omega_k \tag{25}$$

Here x_e is the exact solution of the given nonlinear oscillator.

In the present method, it has been shown that the solution yields the less error in each Iterative step compared to the previous Iterative step and finally $|\Omega_2 - \Omega_e| = |1.251 - 1.253| < \varepsilon$, where ε is a small positive number and A is chosen to be unity. From this, it is clear that the adopted method is convergent.

An Iterative method of the form represented by equation (3) with initial guess given in equation (5) is said to be consistent if

$$\lim_{k \to \infty} |x_k - x_e| = 0 \quad \text{or,} \quad \lim_{k \to \infty} |\Omega_k - \Omega_e| = 0 \tag{26}$$

In the present analysis, we see that

$$\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0, \text{ as } |\Omega_e - \Omega_2| \approx 0. \tag{27}$$

Thus, the consistency of the method is achieved.

6 Conclusion

Rearranging the equation by an iteration technique the approximations to the first to the third frequencies are better than corresponding frequencies, which have been obtained by other techniques. It can be observed that the third approximation provides an excellent result.

Competing Interests

Authors have declared that no competing interests exist.

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