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The Extension of Diagram Group over Semigroup Presentation

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Original Research Article

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Abstract

In this paper, we will discuss the diagram groups from union of two semigroup presentations namely $^2\mathrm{S} = \langle x, y : x = y \rangle$, $^3\mathrm{S} = \langle a, b, c : a = b, b = c, c = a \rangle$ and their two complex graphs will be presented. The covering space will be determined by selecting normal subgroup from diagram group that previously obtained from $^2\mathrm{S} \cup {}^3\mathrm{S}$. Finally, the number of generator and relations of the diagram group can be computed.

Keywords: Generators; relations; diagram groups; semigroup presentation.

1 Introduction

Graph theoretical and geometrical methods have played an important role in the development of semigroup presentation and diagram groups [1-4,5]. This study addresses a new method for studying diagram groups.

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For any given semigroup presentation, $S =< X: R >$, the diagram group $D(S, U)$, where U is a positive word on X [6], can be obtained. The associated group with semigroup presentation is called $K(S)$. For a 2-complex graph, there is a fundamental group $\pi_1(K(S), U)$ with basepoint U. Kilibarda [7,8] showed that the fundamental group is isomorphic to diagram group $D(S, U)$. Therefore, it is sufficient to consider $\pi_1(K(S), U)$ instead of $D(S, U)$. This allows for constructing the fundamental group $\pi_1(K(S), U)$ from the union of two semigroup presentations [9-12].

In fact, Guba and Sapir [6] have shown that if $S_1 =< X_1: R_1 >, S_2 =< X_2: R_2 >$ and $S =< X_1 \cup X_2 : R_1 \cup R_2 >$ are semigroup presentations, then for $U_1, U_2 \in X^+, D(S, U_1U_2)$ is isomorphic to the direct product of $D(S, U_1)$ and $D(S, U_2)$. Also they proved if one consider $S =< X_1 \cup X_2 : R_1 \cup R_2 \cup \{U_1 = U_2\} >$ where X_1, X_2 are disjoint sets, and the congruence class of U_i modulo S_i does not contain words of the form YU_iZ, where Y, Z are words over X_1, X_2 and YZ are not empty, then $D(S, U_i)$ is isomorphic to the free product of $D(S_1, U_i)$ and $D(S_2, U_i)$. Upon that, it is recommended for future research to consider the semigroup presentation $S =< X_1 \cup X_2 : R_1 \cup R_2 \cup \{U_1 = U_2\} >$ for the current method developed in this paper.

In [13] and [14] we obtained the connected 2-complex graphs ${}^2K_i$ and ${}^3K_i, i \in N$ that were obtained from ${}^2S =< x , y : x = y >$, and ${}^3S =< a , b , c : a = b , b = c , c = a >$ respectively.

In this paper we want determine the semigroup presentation of union of two semigroup presentation by adding a relation.

Let ${}^2S =< x , y : x = y >$, ${}^3S =< a , b , c : a = b , b = c , c = a >$ be semigroup presentations. Now we consider the semigroup presentation obtained from union of ${}^2S$ and ${}^3S$ by adding a relation $x = a$.

2 Determining the Two Complex Graphs

In this section all connected two complex graph that are obtained from

$$ {}^5S = {}^2S \cup {}^3S =< x , y , a , b , c : x = y, a = b , b = c , c = a, , x = a > $$

will be constructed.

1. Let $L(U) = 1$, where U is positive words on ${}^5S$. so, the connected two complex graph ${}^5K_1$is given by Fig. 1.

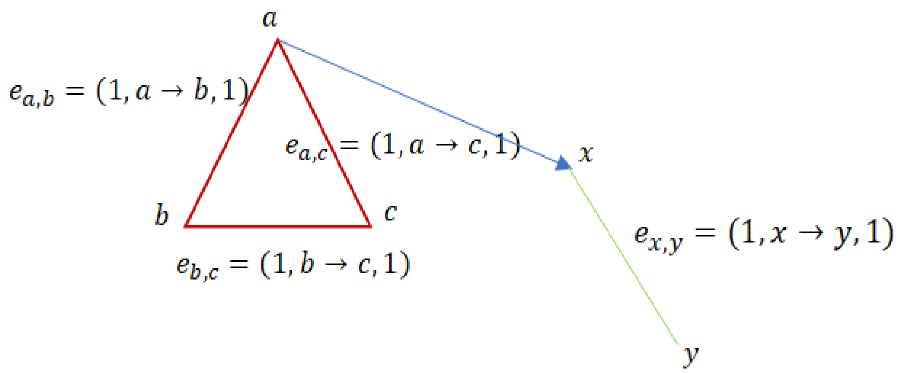

Fig. 1. The connected two complex graph ${}^5K_1$

Note that when $L(U) = 1$, there will be five vertices and five edges in ${}^5K_1$.

2. Let $L(U) = 2$. In this case there are $5^2 = 25$ possibilities vertices in the connected two complex graph ${}^5K_2$(see Fig. 2).

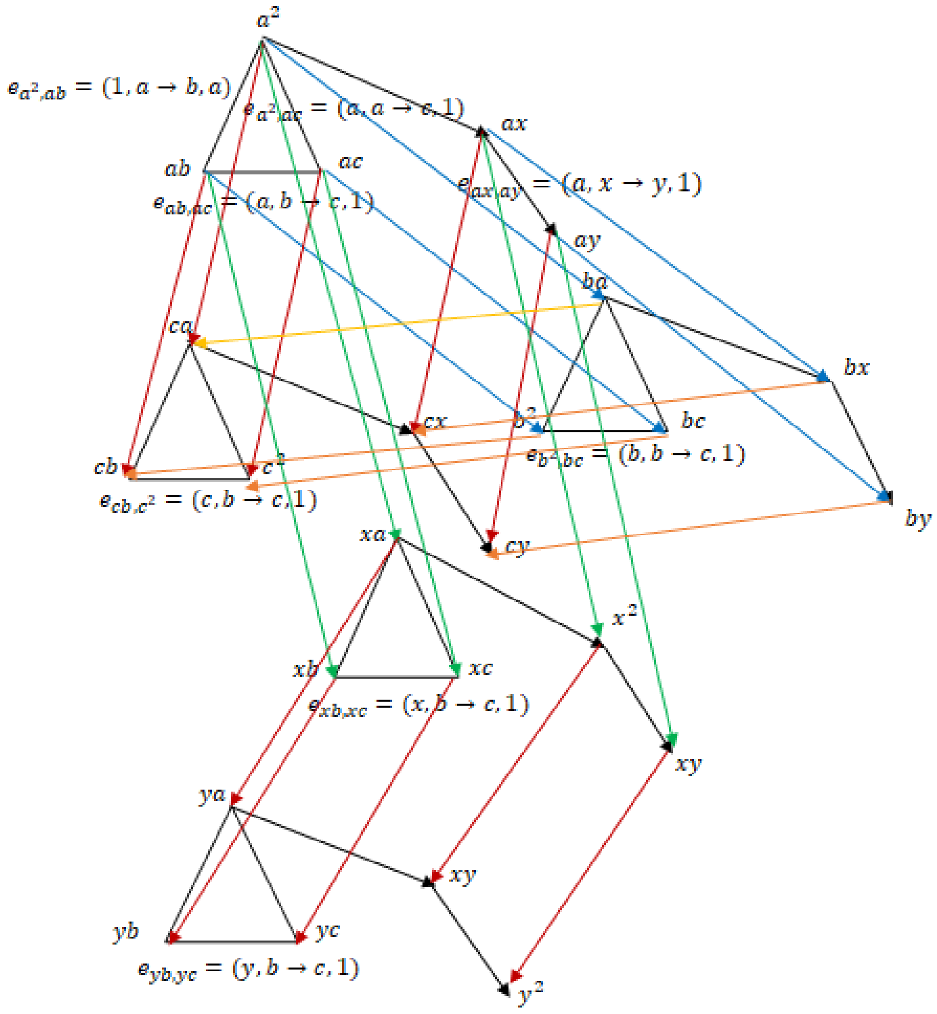

Fig. 2. Connected 2-complex graph $^5K_2$

Corollary 1: A connected 2 —complex graph ^5K_ncontains 5^n vertices.

Corollary 2: Vertices v_1 and v_2 are connected if and only if $L(v_1) = L(v_2)$.

Lemma 3:[15]: If $L((W_1)) = L(W_2))$ then $\pi_1(^5K_n, W_1) = \pi_1(^5K_n, W_2)$.

Lemma 4: Vertices of ^5K_n are all words of length n.

Lemma 5: [16,17]: Let $f : K' \to K$ be a mapping of 2-complexes graphs. If $\tilde{v}$ is a vertex in K', then there is induced monomorphism

$$f^*: \pi_1(K', v') \to \pi_1\big(K, f(v')\big)$$

defined by $f^*[\alpha'] = [f(\alpha')]$.

Lemma 6: [16,17]: The mapping $f^*: \pi_1(K', v') \to \pi_1\big(K, f(v')\big)$ is an injective if f is a locally bijective.

Lemma 7: [16,17]**:** The map $f_N : ^5K_N \to {}^5K$, $f_N(N[\alpha]) = v$, $f_N(N[\alpha], x) = x$ is a mapping of connected 2-complex graphs.

Lemma 8: [16,17]**:** The map $f_N : {}^5K_N \to {}^5K$, $f_N(N[\alpha]) = v$, $f_N(N[\alpha], x) = x$ is locally bijective.

Theorem 1: Consider the following connected two complex graph ${}^5K_1$ as shown in Fig. 1, such that $G = \pi_1({}^5K_1, a)$ contains μ, where $\mu =< e_{a,b}e_{b,c}e_{a,c} >$. If N is the smallest normal subgroup of G containing $< \mu^2 >$, then the covering complex ${}^5(K_N)_1$ for ${}^5K_1$ is a hexagonal shape plus one triangle.

Proof: From ${}^5K_1$, $\pi_1({}^5K_1)$ can be obtained. Fix a vertex a in ${}^5K_1$. Now, for any normal subgroup of $\pi_1({}^5K_1, a)$, there exists a unique covering space. Start by choosing basic $N[\mu]$ where μ is a path such that $i(\mu) = a$, $\tau(\mu) = v$ for every vertex v in ${}^5K_1$. As a result, these basic $N[1]$, $N[e_{a,b}]$, and $N[e_{a,b}e_{b,c}]$ will be designated, and then all possible edges can be determined, as shown in Table 1.

Table 1. Edges from $N[1]$ in ${}^2K_N$

Edges	Initial	Terminal
$(N[1], e_{a,b})$	$N[1]$	$N[e_{a,b}]$
$(N[1], e_{a,b}e_{b,c}e_{a,c}e_{a,b}e_{b,c})$	$N[1]$	$N[e_{a,b}e_{b,c}e_{a,c}e_{a,b}e_{b,c}]$

Since $f_N[N[1]] = a$ and $star(a) = 3$, then $star(N[1]) = 3$. Consider a vertex a; the vertex in ${}^5K_N$ is $N[1]$, and $N[1]$ in ${}^5K_N$ maps to a. From $a \to b$ in ${}^5K_1$, the vertex in ${}^5K_N$ is $N[e_{a,b}]$, and the edge is $(N[1], e_{a,b})$. $N[e_{a,b}]$ in ${}^5K_N$ maps to b in ${}^5K_1$, as shown in Fig. 3.

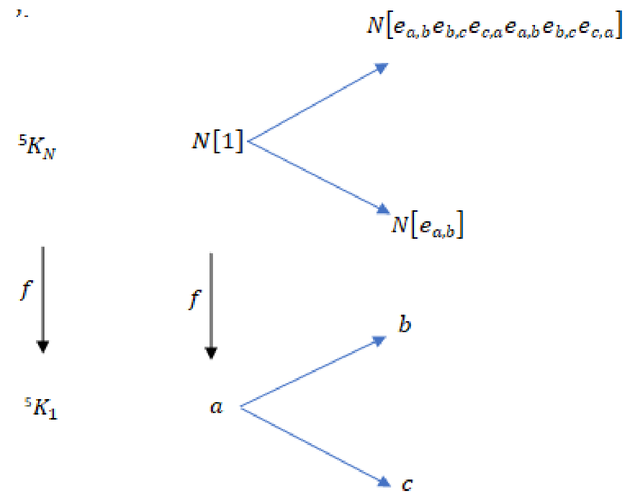

Fig. 3. Mapping from ${}^5(K_N)_1$ to ${}^5K_1$

Similarly, the same applied procedure is used to determine the vertices and the edges.

Table 2 and Table 3 summarize the results of all possible vertices and the edges respectively.

Table 2. Vertices in ${}^5K_1$and ${}^5(K_N)_1$

Vertex in ${}^5K_1$	Vertex v in ${}^5(K_N)_1$
a	$N[1]$
b	$N[e_{a,b}]$
c	$N[e_{a,b}e_{b,c}]$
a	$N[e_{a,b}e_{b,c}e_{c,a}]$
b	$N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}]$
c	$N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c}]$
x	$N[e_{a,x}]$
y	$N[e_{a,x}e_{x,y}]$

Table 3. Edges in $^5K_1$and $^5(K_N)_1$

Edges in $^5K_1$	Edges in $^5(K_N)_1$
$e_{a,b}$	$(N[1]\,, e_{a,b})$
$e_{a,b}e_{b,c}$	$(N[e_{a,b}]\,, e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(N[e_{a,b}e_{b,c}], e_{a,b}e_{b,c}e_{c,a})$
$e_{a,b}$	$(N[e_{a,b}e_{b,c}e_{c,a}], e_{a,b}e_{b,c}e_{c,a}e_{a,b})$
$e_{a,b}e_{b,c}$	$(N[e_{a,b}e_{b,c}e_{c,a}e_{a,b}], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,b}e_{b,c}e_{c,a}$	$(N[1], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c})$
$e_{a,x}$	$(N[1]\,, e_{a,x})$
$e_{a,x}e_{x,y}$	$(N[e_{a,x}]\,, e_{a,x}e_{x,y})$

Now suppose $f_N : {}^5(K_N)_1 \to {}^5K_1$defined by $f_N(N[1]) = a\,, f_N(N[e_{a,x}]) = x, f_N(N[\alpha], e_{a,x}) = e_{a,x}$. This map can be viewed as locally bijective. For this reason, $^5(K_N)_1$ is the covering space for $^5K_1$ and it is of hexagonal shape plus one triangle. Therefore, the covering space $^5(K_N)_1$for $^5K_1$in this case is of hexagonal shape plus one triangle, as shown in Fig. 4.

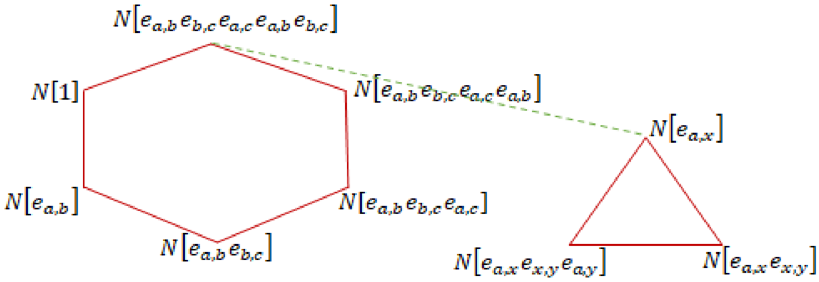

Fig. 4. Covering complex $^5(K_N)_1$

Since a is a vertex of the connected two complex $^5K_1$, and $N[1]$ lies over a, then by LEMMA 6, $f_N^*: \pi_1(^5(K_N)_1, N[1]) \to \pi_1(^5K_1, a)$ is injective. Therefore, $f_N^*: \pi_1(^5(K_N)_1, N[1]) \to Imf_N^* = N$. As a result, $N\pi_1(^5(K_N)_1, N[1])$ can be considered as a subgroup of $G = \pi_1(^5K_1, a)$.

The generators for $\pi_1(^5(K_N)_1, N[1])$ are computed here using maximal subtree methods. Select a maximal subtree T(^5K_N) for $^5(K_N)_1$ (see Fig. 5).

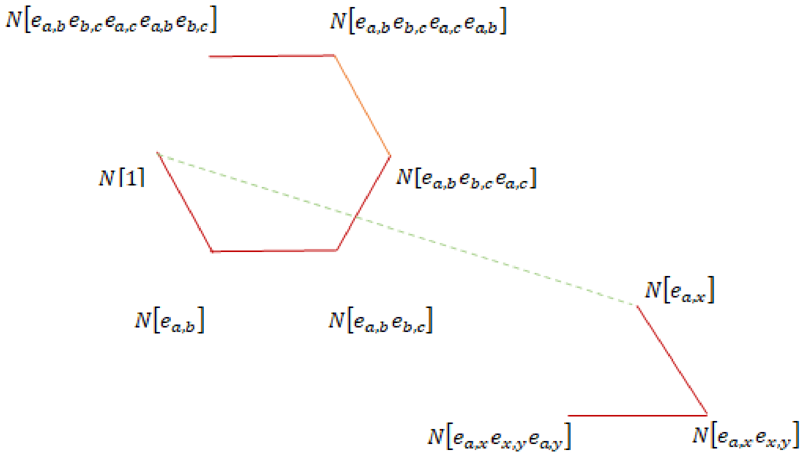

Fig. 5. Maximal subtree T($^5(K_N)_1$)

The generators for the fundamental group $\pi_1(^5(K_N)_1, N[1])$ will be:

$$g_1(5K_N) = \left(N[1]\, , e_{a,b}\right)\left(N\left[e_{a,b}\right]\, , e_{a,b}e_{b,c}\right)\left(N\left[e_{a,b}e_{b,c}\right], e_{a,b}e_{b,c}e_{c,a}\right)$$
$$\left(N\left[e_{a,b}e_{b,c}e_{c,a}\right], e_{a,b}e_{b,c}e_{c,a}e_{a,b}\right)\left(N\left[e_{a,b}e_{b,c}e_{c,a}e_{a,b}\right], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c}\right)\left(N[1], e_{a,b}e_{b,c}e_{c,a}e_{a,b}e_{b,c}\right)^{-1}.$$
$$g_2(\pi_1(^5K_N)) = \left(N[1], e_{a,x}\right)\left(N\left[e_{a,x}\right], e_{a,x}e_{x,y}\right)\left(N\left[e_{a,x}e_{x,y}\right], e_{a,x}e_{x,y}e_{a,y}\right)$$
$$\left(N\left[e_{a,x}\right], e_{a,x}e_{x,y}\right)^{-1}\left(N[1], e_{a,x}\right)^{-1}.$$

Theorem 2: Let the following semigroup presentation

$$^5S = < x\, , y\, , a\, , b\, , c\ :\ x = y, a = b\, , b = c\, , c = a,\ \ , x = a >$$

If the number of all vertices of two complex graph ^5K_N is 5^n, then the number of all vertices of the covering space $^5(K_N)_n$ is $(5)^n + 3$.

Proof: By induction, for $k = 1$ the number of all vertices in $^5(K_N)_1$ is 5. Thus for $k = 1$ is true (see Fig. 1). Now assume $v_k = (5)^k + 3$ be the number of all vertices in $^5(K_N)_k$.

We will prove the number of all vertices of the covering space $^5(K_N)_{k+1}$ is $(5)^{k+1} + 3$. By the definition of $^5K_{k+1}$ is five copies of ^5K_k and assumption, then the number of all vertices of the covering space $^5(K_N)_{k+1}$ is $v_{k+1} = 5 \cdot (5)^k + 3 = (5)^{k+1} + 3$.

Theorem 3: Consider the semigroup presentation

$$^5S = < x\, , y\, , a\, , b\, , c\ :\ x = y, a = b\, , b = c\, , c = a,\ \ , x = a >.$$

The number of all edges in the covering space $^5(K_N)_n$ is $e_n = n5^n + 3$.

Proof: By induction, for $k = 1$ the number of all vertices in $^5(K_N)_1$ is $e_1 = 1(5) + 3$=8.

Now let $e_k = k5^k + 3$ be the number of all edges the covering space $^5(K_N)_k$. We will prove that the number of all edges in $^5(K_N)_{k+1}$ is $e_{k+1} = (k + 1)(5)^{k+1} + 3$. By using last theorem

$$e_{k+1} = 5e_k + 5^{k+1} + 3 = 5k5^k + 5^{k+1} + 3 = k \cdot 5^{k+1} + 5^{k+1} + 3 = (k + 1)5^{k+1} + 3\ .$$

3 Conclusion

The paper provided, a new technique which has been explored to study diagram groups that was previously obtained from a union of two semigroup presentations

$$^5S = {^2S} \cup {^3S} = < x\, , y\, , a\, , b\, , c\ :\ x = y, a = b\, , b = c\, , c = a,\ \ , x = a >$$

By adding a relation.

The paper discussed how to determine the covering complex $^5(K_N)_1$for the connected two complex graph $^5K_1$ by selection normal subgroup from the diagram group. Also, this paper discussed how the generators and the relations for the fundamental group $\pi_1(^5(K_N)_1, N[1])$ were calculated by using maximal tree methods. Finally, the number of all vertices and edges in the covering space $^5(K_N)_1$were computed.

Competing Interests

Author has declared that no competing interests exist.

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