On the Sums of Squares of Generalized Tribonacci Numbers: Closed Formulas of $\sum_{k=0}^{n} x^k W_k^2$

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ABSTRACT

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_k^2$, $\sum_{k=0}^{n} x^k W_{k+1} W_k$ and $\sum_{k=0}^{n} x^k W_{k+2} W_k$ for the squares of generalized Tribonacci numbers are presented. As special cases, we give summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin numbers and the other third order recurrence relations. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery. Our work generalize third order recurrence relations.

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1   INTRODUCTION

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n \geq 0}$ (or shortly $\{W_n\}_{n \geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \; n \geq 3 \quad\quad (1.1)$$

where W_0, W_1, W_2 are arbitrary complex numbers and r, s, t are real numbers. The generalized Tribonacci sequence has been studied by many authors, see for example [1-15].

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t and initial values.

Table 1. A few special case of generalized Tribonacci sequences

Sequences (Numbers)	Notation	OEIS [16]
Tribonacci	$\{T_n\} = \{V_n(0, 1, 1; 1, 1, 1)\}$	A000073, A057597
Tribonacci-Lucas	$\{K_n\} = \{V_n(3, 1, 3; 1, 1, 1)\}$	A001644, A073145
third order Pell	$\{P_n^{(3)}\} = \{V_n(0, 1, 2; 2, 1, 1)\}$	A077939, A077978
third order Pell-Lucas	$\{Q_n^{(3)}\} = \{V_n(3, 2, 6; 2, 1, 1)\}$	A276225, A276228
third order modified Pell	$\{E_n^{(3)}\} = \{V_n(0, 1, 1; 2, 1, 1)\}$	A077997, A078049
Padovan (Cordonnier)	$\{P_n\} = \{V_n(1, 1, 1; 0, 1, 1)\}$	A000931
Perrin (Padovan-Lucas)	$\{E_n\} = \{V_n(3, 0, 2; 0, 1, 1)\}$	A001608, A078712
Padovan-Perrin	$\{S_n\} = \{V_n(0, 0, 1; 0, 1, 1)\}$	A000931, A176971
Pell-Padovan	$\{R_n\} = \{V_n(1, 1, 1; 0, 2, 1)\}$	A066983, A128587
Pell-Perrin	$\{C_n\} = \{V_n(3, 0, 2; 0, 2, 1)\}$	
Jacobsthal-Padovan	$\{Q_n\} = \{V_n(1, 1, 1; 0, 1, 2)\}$	A159284
Jacobsthal-Perrin (-Lucas)	$\{L_n\} = \{V_n(3, 0, 2; 0, 1, 2)\}$	A072328
Narayana	$\{N_n\} = \{V_n(0, 1, 1; 1, 0, 1)\}$	A078012
Narayana-Lucas	$\{U_n\} = \{V_n(3, 1, 1; 1, 0, 1)\}$	A001609
Narayana-Perrin	$\{H_n\} = \{V_n(3, 0, 2; 1, 0, 1)\}$	
third order Jacobsthal	$\{J_n^{(3)}\} = \{V_n(0, 1, 1; 1, 1, 2)\}$	A077947
third order Jacobsthal-Lucas	$\{j_n^{(3)}\} = \{V_n(2, 1, 5; 1, 1, 2)\}$	A226308
3-primes	$\{G_n\} = \{V_n(0, 1, 2; 2, 3, 5)\}$	
Lucas 3-primes	$\{H_n\} = \{V_n(3, 2, 10; 2, 3, 5)\}$	
modified 3-primes	$\{E_n\} = \{V_n(0, 1, 1; 2, 3, 5)\}$	

Here OEIS stands for On-line Encyclopedia of Integer Sequences. 3-primes, Lucas 3-primes and modified 3-primes sequences can also be called (named) as Grahaml, Grahaml-Lucas and modified Grahaml sequences, respectively, see [17].

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=0}^{n} (-1)^k T_k^2 = \frac{1}{4}((-1)^n \left(T_{n+3}^2 - 2T_{n+2}^2 + 3T_{n+1}^2 - 2T_{n+1}T_{n+3}\right) - 1)$$

and

$$\sum_{k=0}^{n}(-1)^k N_k^2 = \frac{1}{3}((-1)^n \left(N_{n+3}^2 - 2N_{n+2}^2 + 2N_{n+1}^2 - 2N_{n+3}N_{n+1} + 2N_{n+2}N_{n+1}\right) - 1).$$

In this work, we derive expressions for sums of second powers of generalized Tribonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[18,19,20,21,22,23]	[24,25,26,27,28]	[29,30,31]
Generalized Tribonacci	[32,33]		
Generalized Tetranacci	[34,35]		

2 AN APPLICATION OF THE SUM OF THE SQUARES OF THE NUMBERS

An application of the sum of the squares of the numbers is circulant matrix. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. For generalized m-step Fibonacci sequences see for example Soykan [36]. If $m = 2, m = 3$ and $m = 4$, we get the generalized Fibonacci sequence, generalized Tribonacci sequence and generalized Tetranacci sequence, respectively.

Next, we recall some information on circulant (r-circulant, geometric circulant) matrices and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm.

Circulant matrices have been around for a long time and have been extensively used in many scientific areas. In some scientific areas such as image processing, coding theory and signal processing we often encounter circulant matrices. These matrices also have many applications in numerical analysis, optimization, digital image processing, mathematical statistics and modern technology.

Let $n \geq 2$ be an integer and r be any real or complex number. An $n \times n$ matrix C_r is called a r-circulant matrix if it of the form

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n \times n}.$$

and the r-circulant matrix C_r is denoted by $C_r = Circ_r(c_0, c_1, ..., c_{n-1})$. If $r = 1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = Circ(c_0, c_1, ..., c_{n-1})$.

- Circulant matrix was first proposed by Davis in [37]. This matrix has many interesting properties, and it is one of the most important research subject in the field of the computational and pure mathematics (see for example references given in Table 3). For instance, Shen and Cen [38] studied on the norms of r-circulant matrices with Fibonacci and Lucas numbers.

- Then, later Kızılateş and Tuglu [39] defined a new geometric circulant matrix as follows:

$$
C_{r^*} = \begin{pmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
r^2 c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
r^{n-1} c_1 & r^{n-2} c_2 & r^{n-3} c_3 & \cdots & rc_{n-1} & c_0
\end{pmatrix}_{n \times n}.
$$

  and then they obtained the bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci number and Lucas numbers.

- When the parameter satisfies $r = 1$, we get the classical circulant matrix. See also Polatlı [40] for the spectral norms of r-circulant matrices with a type of Catalan triangle numbers.

The Frobenius (or Euclidean) norm and spectral norm of a matrix $A = (a_{ij})_{m \times n} \in M_{m \times n}(\mathbb{C})$ are defined respectively as follows:

$$
\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad \|A\|_2 = \left(\max_{1 \le i \le n} |\lambda_i| \right)^{1/2}
$$

where λ_i's are the eigenvalues of the matrix $A^* A$ and A^* is the conjugate of transpose of the matrix A. The maximum column length norm $c_1(.)$ and the maximum row length norm $r_1(.)$ of an matrix of order $n \times n$ are defined as follows:

$$
c_1(A) = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad r_1(A) = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.
$$

The following inequality holds for any matrix $A = M_{n \times n}(\mathbb{C})$:

$$
\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F.
$$

Calculations of the above norms $\|A\|_F$, $\|A\|_2$, $c_1(A)$ and $r_1(A)$ require the sum of the squares of the numbers a_{ij}. As in our case, the numbers a_{ij} can be chosen as elements of second, third or higher order linear recurrence sequences.

In the following Table 3, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences which require sum formulas of second powers of numbers in m-step Fibonacci sequences ($m = 2, 3, 4$).

Table 3. Papers on the norms

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[41,39,42,43,44,45,46,38,47,48,49,50]
Pell, Pell-Lucas	[51,52]
Jacobsthal, Jacobsthal-Lucas	[53,54,55,56]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[57,58]
Padovan, Perrin	[59,60,61]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[62]

Also linear summing formulas of the generalized m-step Fibonacci sequences are required for the computation of various norms of circulant matrices circulant matrices with the generalized m-step Fibonacci sequences. We present some works on summing formulas of the numbers in the following Table 4.

Table 4. A few special study of sum formulas

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[63],[64,65]
Generalized Fibonacci	[66,67,68,69,70]
Generalized Tribonacci	[71,72,73]
Generalized Tetranacci	[74,75,76]
Generalized Pentanacci	[77,78]
Generalized Hexanacci	[79]

3 MAIN RESULT

Theorem 3.1. *Let x be a complex number. If $\Delta = (-t^2x^3 + sx + rtx^2 + 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) \neq 0$ then*

(a)

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta_1}{(-t^2x^3 + sx + rtx^2 + 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1)},$$

(b)

$$\sum_{k=0}^{n} x^k W_{k+1} W_k = \frac{\Delta_2}{(-t^2x^3 + sx + rtx^2 + 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1)},$$

(c)

$$\sum_{k=0}^{n} x^k W_{k+2} W_k = \frac{\Delta_3}{(-t^2x^3 + sx + rtx^2 + 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1)},$$

where

$$
\begin{aligned}
\Delta_1 &= -x^{n+3}(t^2x^3 + sx + rtx^2 - 1)W_{n+3}^2 \\
&\quad -x^{n+2}(r^2x + t^2x^3 + sx + r^2t^2x^4 + rtx^2 + r^2sx^2 + r^3tx^3 + 2rstx^3 - 1)W_{n+2}^2 \\
&\quad -x^{n+1}(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + r^2t^2x^4 \\
&\quad +s^2t^2x^5 + rtx^2 + r^2sx^2 + r^3tx^3 + 4rstx^3 - rs^2tx^4 - 1)W_{n+1}^2 \\
&\quad + + x^2(t^2x^3 + sx + rtx^2 - 1)W_2^2 \\
&\quad +x(r^2x + t^2x^3 + sx + r^2t^2x^4 + rtx^2 + r^2sx^2 + r^3tx^3 + 2rstx^3 - 1)W_1^2 \\
&\quad +(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + r^2t^2x^4 + s^2t^2x^5 + rtx^2 \\
&\quad +r^2sx^2 + r^3tx^3 + 4rstx^3 - rs^2tx^4 - 1)W_0^2 \\
&\quad +2x^{n+4}(r + tx)(s + rtx)W_{n+3}W_{n+2} + 2x^{n+4}t(r + stx^2)W_{n+3}W_{n+1} \\
&\quad -2x^{n+4}t(sx - 1)(s + rtx)W_{n+2}W_{n+1} - 2x^3(r + tx)(s + rtx)W_2W_1 \\
&\quad -2tx^3(r + stx^2)W_2W_0 + 2x^3t(sx - 1)(s + rtx)W_1W_0
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_2 \\
= \quad & x^{n+3}(r + stx^2)W_{n+3}^2 + x^{n+4}(t + rs)(s + rtx)W_{n+2}^2 + x^{n+4}t^2(r + stx^2)W_{n+1}^2 \\
& -x^{n+2}(r^2x + s^2x^2 + t^2x^3 + 2rstx^3 - 1)W_{n+3}W_{n+2} \\
& +x^{n+3}t(r^2x - s^2x^2 - t^2x^3 + 1)W_{n+3}W_{n+1} \\
& -x^{n+1}(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + rtx^2 + r^2sx^2 + r^3tx^3 \\
& -rt^3x^5 - st^2x^4 + 2rstx^3 - rs^2tx^4 - 1)W_{n+2}W_{n+1} - x^2(r + stx^2)W_2^2 \\
& -x^3(t + rs)(s + rtx)W_1^2 - x^3t^2(r + stx^2)W_0^2 + x(r^2x + s^2x^2 + t^2x^3 + 2rstx^3 - 1)W_2W_1 \\
& -x^2t(r^2x - s^2x^2 - t^2x^3 + 1)W_2W_0 + (r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx \\
& +rtx^2 + r^2sx^2 + r^3tx^3 - rt^3x^5 - st^2x^4 + 2rstx^3 - rs^2tx^4 - 1)W_1W_0
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_3 = \quad & x^{n+3}(s - s^2x + r^2 + rtx)W_{n+3}^2 + x^{n+2}(s - s^2x + r^2t^2x^3 - r^2sx + rt^3x^4 - rs^2tx^3)W_{n+2}^2 \\
& +x^{n+4}t^2(s - s^2x + r^2 + rtx)W_{n+1}^2 - x^{n+2}(r + tx)(r^2x - s^2x^2 + t^2x^3 - 1)W_{n+3}W_{n+2} \\
& -x^{n+1}(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + r^2sx^2 - st^2x^4 + 2rstx^3 - 1)W_{n+3}W_{n+1} \\
& +x^{n+2}t(sx - 1)(r^2x - s^2x^2 + t^2x^3 - 1)W_{n+2}W_{n+1} - x^2(s - s^2x + r^2 + rtx)W_2^2 \\
& +x(-s + s^2x - r^2t^2x^3 + r^2sx - rt^3x^4 + rs^2tx^3)W_1^2 - x^3t^2(s - s^2x + r^2 + rtx)W_0^2 \\
& +x(r + tx)(r^2x - s^2x^2 + t^2x^3 - 1)W_2W_1 \\
& +(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + r^2sx^2 - st^2x^4 + 2rstx^3 - 1)W_2W_0 \\
& -xt(sx - 1)(r^2x - s^2x^2 + t^2x^3 - 1)W_1W_0.
\end{aligned}
$$

Proof. First, we obtain $\sum_{k=0}^{n} W_k^2$. Using the recurrence relation

$$
W_{n+3} = rW_{n+2} + sW_{n+1} + tW_n
$$

or

$$
tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}
$$

i.e.

$$
t^2 W_n^2 = (W_{n+3} - rW_{n+2} - sW_{n+1})^2 = W_{n+3}^2 + r^2 W_{n+2}^2 + s^2 W_{n+1}^2 - 2rW_{n+3}W_{n+2} - 2sW_{n+3}W_{n+1} + 2rsW_{n+2}W_{n+1}
$$

we obtain

$$
\begin{aligned}
t^2 x^n W_n^2 \quad = \quad & x^n W_{n+3}^2 + r^2 x^n W_{n+2}^2 + s^2 x^n W_{n+1}^2 - 2rx^n W_{n+3}W_{n+2} - 2sx^n W_{n+3}W_{n+1} + 2rsx^n W_{n+2}W_{n+1} \\
t^2 x^{n-1} W_{n-1}^2 \quad = \quad & x^{n-1} W_{n+2}^2 + r^2 x^{n-1} W_{n+1}^2 + s^2 x^{n-1} W_n^2 - 2rx^{n-1} W_{n+2}W_{n+1} \\
& -2sx^{n-1} W_{n+2}W_n + 2rsx^{n-1} W_{n+1}W_n \\
t^2 x^{n-2} W_{n-2}^2 \quad = \quad & x^{n-2} W_{n+1}^2 + r^2 x^{n-2} W_n^2 + s^2 x^{n-2} W_{n-1}^2 - 2rx^{n-2} W_{n+1}W_n \\
& -2sx^{n-2} W_{n+1}W_{n-1} + 2rsx^{n-2} W_n W_{n-1} \\
& \quad\vdots \\
t^2 x^2 W_2^2 \quad = \quad & x^2 W_5^2 + r^2 x^2 W_4^2 + s^2 x^2 W_3^2 - 2rx^2 W_5 W_4 - 2sx^2 W_5 W_3 + 2rsx^2 W_4 W_3 \\
t^2 x^1 W_1^2 \quad = \quad & x^1 W_4^2 + r^2 x^1 W_3^2 + s^2 x^1 W_2^2 - 2rx^1 W_4 W_3 - 2sx^1 W_4 W_2 + 2rsx^1 W_3 W_2 \\
t^2 x^0 W_0^2 \quad = \quad & x^0 W_3^2 + r^2 x^0 W_2^2 + s^2 x^0 W_1^2 - 2rx^0 W_3 W_2 - 2sx^0 W_3 W_1 + 2rsx^0 W_2 W_1
\end{aligned}
$$

If we add the equations by side by, we get

$$
\begin{aligned}
t^2 \sum_{k=0}^{n} x^k W_k^2 \quad = \quad & \sum_{k=3}^{n+3} x^{k-3} W_k^2 + r^2 \sum_{k=2}^{n+2} x^{k-2} W_k^2 + s^2 \sum_{k=1}^{n+1} x^{k-1} W_k^2 \qquad\qquad (3.1) \\
& -2r \sum_{k=2}^{n+2} x^{k-2} W_{k+1} W_k - 2s \sum_{k=1}^{n+1} x^{k-1} W_{k+2} W_k + 2rs \sum_{k=1}^{n+1} x^{k-1} W_{k+1} W_k
\end{aligned}
$$

Next we obtain $\sum_{k=0}^{n} W_{k+1}W_k$. Multiplying the both side of the recurrence relation

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

by W_{n+1} we get

$$tW_{n+1}W_n = W_{n+3}W_{n+1} - rW_{n+2}W_{n+1} - sW_{n+1}^2.$$

Then using last recurrence relation, we obtain

$$
\begin{array}{rcl}
tx^n W_{n+1}W_n & = & x^n W_{n+3}W_{n+1} - rx^n W_{n+2}W_{n+1} - sx^n W_{n+1}^2 \\
tx^{n-1} W_n W_{n-1} & = & x^{n-1} W_{n+2}W_n - rx^{n-1} W_{n+1}W_n - sx^{n-1} W_n^2 \\
tx^{n-2} W_{n-1}W_{n-2} & = & x^{n-2} W_{n+1}W_{n-1} - rx^{n-2} W_n W_{n-1} - sx^{n-2} W_{n-1}^2 \\
& \vdots & \\
tx^2 W_3 W_2 & = & x^2 W_5 W_3 - rx^2 W_4 W_3 - sx^2 W_3^2 \\
tx W_2 W_1 & = & x W_4 W_2 - rx W_3 W_2 - sx W_2^2 \\
tx^0 W_1 W_0 & = & x^0 W_3 W_1 - rx^0 W_2 W_1 - sx^0 W_1^2
\end{array}
$$

If we add the equations by side by, we get

$$t \sum_{k=0}^{n} x^k W_{k+1}W_k = \sum_{k=1}^{n+1} x^{k-1} W_{k+2}W_k - r \sum_{k=1}^{n+1} x^{k-1} W_{k+1}W_k - s \sum_{k=1}^{n+1} x^{k-1} W_k^2. \qquad (3.2)$$

Next we obtain $\sum_{k=2}^{n} W_{k+2}W_k$. Multiplying the both side of the recurrence relation

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

by W_{n+2} we get

$$tW_{n+2}W_n = W_{n+3}W_{n+2} - rW_{n+2}^2 - sW_{n+2}W_{n+1}.$$

Then using last recurrence relation, we obtain

$$
\begin{array}{rcl}
tx^n W_{n+2}W_n & = & x^n W_{n+3}W_{n+2} - rx^n W_{n+2}^2 - sx^n W_{n+2}W_{n+1} \\
tx^{n-1} W_{n+1}W_{n-1} & = & x^{n-1} W_{n+2}W_{n+1} - rx^{n-1} W_{n+1}^2 - sx^{n-1} W_{n+1}W_n \\
tx^{n-2} W_n W_{n-2} & = & x^{n-2} W_{n+1}W_n - rx^{n-2} W_n^2 - sx^{n-2} W_n W_{n-1} \\
& \vdots & \\
tx^2 W_4 W_2 & = & x^2 W_5 W_4 - rx^2 W_4^2 - sx^2 W_4 W_3 \\
tx^1 W_3 W_1 & = & x^1 W_4 W_3 - rx^1 W_3^2 - sx^1 W_3 W_2 \\
tx^0 W_2 W_0 & = & x^0 W_3 W_2 - rx^0 W_2^2 - sx^0 W_2 W_1
\end{array}
$$

If we add the equations by side by, we get

$$t \sum_{k=0}^{n} x^k W_{k+2}W_k = \sum_{k=2}^{n+2} x^{k-2} W_{k+1}W_k - r \sum_{k=2}^{n+2} x^{k-2} W_k^2 - s \sum_{k=1}^{n+1} x^{k-1} W_{k+1}W_k \qquad (3.3)$$

Solving the system (3.1)-(3.2)-(3.3), the results in (a), (b) and (c) follow.

4 SPECIFIC CASES

In this section, for the specific cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n} x^k W_k^2$, $\sum_{k=0}^{n} x^k W_{k+1}W_k$ and $\sum_{k=0}^{n} x^k W_{k+2}W_k$ for the specific case of sequence $\{W_n\}$.

4.1 The case $x = 1$

In this subsection we consider the special case $x = 1$. See also [33] for some third order recurrence relations (with the sum starting from 0).

Taking $r = s = t = 1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 4.1. If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{4}(-W_{n+3}^2 - 4W_{n+2}^2 - 5W_{n+1}^2 + 4W_{n+3}W_{n+2} + 2W_{n+3}W_{n+1} + W_2^2 + 4W_1^2 + 5W_0^2 - 4W_2W_1 - 2W_2W_0)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{4}(W_{n+3}^2 + 2W_{n+2}^2 + W_{n+1}^2 - 2W_{n+3}W_{n+2} - 2W_{n+2}W_{n+1} - W_2^2 - 2W_1^2 - W_0^2 + 2W_2W_1 + 2W_1W_0)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{4}(W_{n+3}^2 + W_{n+1}^2 - 2W_{n+3}W_{n+1} - W_2^2 - W_0^2 + 2W_2W_0)$.

From the above proposition, we have the following Corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 4.2. For $n \geq 0$, Tribonacci numbers have the following properties:

(a) $\sum_{k=0}^{n} T_k^2 = \frac{1}{4}(-T_{n+3}^2 - 4T_{n+2}^2 - 5T_{n+1}^2 + 4T_{n+3}T_{n+2} + 2T_{n+3}T_{n+1} + 1)$.

(b) $\sum_{k=0}^{n} T_{k+1}T_k = \frac{1}{4}(T_{n+3}^2 + 2T_{n+2}^2 + T_{n+1}^2 - 2T_{n+3}T_{n+2} - 2T_{n+2}T_{n+1} - 1)$.

(c) $\sum_{k=0}^{n} T_{k+2}T_k = \frac{1}{4}(T_{n+3}^2 + T_{n+1}^2 - 2T_{n+3}T_{n+1} - 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above Proposition, we have the following Corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 4.3. For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} K_k^2 = \frac{1}{4}(-K_{n+3}^2 - 4K_{n+2}^2 - 5K_{n+1}^2 + 4K_{n+3}K_{n+2} + 2K_{n+3}K_{n+1} + 28)$.

(b) $\sum_{k=0}^{n} K_{k+1}K_k = \frac{1}{4}(K_{n+3}^2 + 2K_{n+2}^2 + K_{n+1}^2 - 2K_{n+3}K_{n+2} - 2K_{n+2}K_{n+1} - 8)$.

(c) $\sum_{k=0}^{n} K_{k+2}K_k = \frac{1}{4}(K_{n+3}^2 + K_{n+1}^2 - 2K_{n+3}K_{n+1})$.

Taking $r = 2, s = 1, t = 1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 4.4. If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{9}(-W_{n+3}^2 - 10W_{n+1}^2 - 9W_{n+2}^2 + 2W_{n+3}W_{n+1} + 6W_{n+3}W_{n+2} + W_2^2 + 9W_1^2 + 10W_0^2 - 6W_2W_1 - 2W_2W_0)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{9}(W_{n+3}^2 + 3W_{n+2}^2 + W_{n+1}^2 - 3W_{n+3}W_{n+2} + W_{n+3}W_{n+1} - 6W_{n+2}W_{n+1} - W_2^2 - 3W_1^2 - W_0^2 + 3W_2W_1 - W_2W_0 + 6W_1W_0)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{9}(2W_{n+3}^2 + 2W_{n+1}^2 - 3W_{n+3}W_{n+2} - 4W_{n+3}W_{n+1} - 2W_2^2 - 2W_0^2 + 3W_2W_1 + 4W_2W_0)$.

From the last Proposition, we have the following Corollary which gives sum formulas of Third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1$).

Corollary 4.5. For $n \geq 0$, third-order Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} P_k^2 = \frac{1}{9}(-P_{n+3}^2 - 10P_{n+1}^2 - 9P_{n+2}^2 + 2P_{n+3}P_{n+1} + 6P_{n+3}P_{n+2} + 1)$.

(b) $\sum_{k=0}^{n} P_{k+1}P_k = \frac{1}{9}(P_{n+3}^2 + 3P_{n+2}^2 + P_{n+1}^2 - 3P_{n+3}P_{n+2} + P_{n+3}P_{n+1} - 6P_{n+2}P_{n+1} - 1)$.

(c) $\sum_{k=0}^{n} P_{k+2}P_k = \frac{1}{9}(2P_{n+3}^2 + 2P_{n+1}^2 - 3P_{n+3}P_{n+2} - 4P_{n+3}P_{n+1} - 2)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last Proposition, we have the following Corollary which presents sum formulas of third-order Pell-Lucas numbers.

Corollary 4.6. *For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} Q_k^2 = \frac{1}{9}(-Q_{n+3}^2 - 10Q_{n+1}^2 - 9Q_{n+2}^2 + 2Q_{n+3}Q_{n+1} + 6Q_{n+3}Q_{n+2} + 54)$.

(b) $\sum_{k=0}^{n} Q_{k+1}Q_k = \frac{1}{9}(Q_{n+3}^2 + 3Q_{n+2}^2 + Q_{n+1}^2 - 3Q_{n+3}Q_{n+2} + Q_{n+3}Q_{n+1} - 6Q_{n+2}Q_{n+1} - 3)$.

(c) $\sum_{k=0}^{n} Q_{k+2}Q_k = \frac{1}{9}(2Q_{n+3}^2 + 2Q_{n+1}^2 - 3Q_{n+3}Q_{n+2} - 4Q_{n+3}Q_{n+1} + 18)$.

From the last Proposition, we have the following Corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 4.7. *For $n \geq 0$, third-order modified Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} E_k^2 = \frac{1}{9}(-E_{n+3}^2 - 10E_{n+1}^2 - 9E_{n+2}^2 + 2E_{n+3}E_{n+1} + 6E_{n+3}E_{n+2} + 4)$.

(b) $\sum_{k=0}^{n} E_{k+1}E_k = \frac{1}{9}(E_{n+3}^2 + 3E_{n+2}^2 + E_{n+1}^2 - 3E_{n+3}E_{n+2} + E_{n+3}E_{n+1} - 6E_{n+2}E_{n+1} - 1)$.

(c) $\sum_{k=0}^{n} E_{k+2}E_k = \frac{1}{9}(2E_{n+3}^2 + 2E_{n+1}^2 - 3E_{n+3}E_{n+2} - 4E_{n+3}E_{n+1} + 1)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 4.8. *If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^2 = -W_{n+3}^2 - W_{n+2}^2 - 2W_{n+1}^2 + 2W_{n+3}W_{n+1} + 2W_{n+3}W_{n+2} + W_2^2 + W_1^2 + 2W_0^2 - 2W_2W_1 - 2W_2W_0$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - W_{n+3}W_{n+2} - W_{n+3}W_{n+1} - W_2^2 - W_1^2 - W_0^2 + W_2W_1 + W_2W_0$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = W_{n+3}W_{n+2} - W_2W_1$.

From the last Proposition, we have the following Corollary which gives sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

Corollary 4.9. *For $n \geq 0$, Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} P_k^2 = -P_{n+3}^2 - P_{n+2}^2 - 2P_{n+1}^2 + 2P_{n+3}P_{n+1} + 2P_{n+3}P_{n+2}$.

(b) $\sum_{k=0}^{n} P_{k+1}P_k = P_{n+3}^2 + P_{n+2}^2 + P_{n+1}^2 - P_{n+3}P_{n+2} - P_{n+3}P_{n+1} - 1$.

(c) $\sum_{k=0}^{n} P_{k+2}P_k = P_{n+3}P_{n+2} - 1$.

Taking $W_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in the last Proposition, we have the following Corollary which presents sum formulas of Perrin numbers.

Corollary 4.10. *For $n \geq 0$, Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} E_k^2 = -E_{n+3}^2 - E_{n+2}^2 - 2E_{n+1}^2 + 2E_{n+3}E_{n+1} + 2E_{n+3}E_{n+2} + 10$.

(b) $\sum_{k=0}^{n} E_{k+1}E_k = E_{n+3}^2 + E_{n+2}^2 + E_{n+1}^2 - E_{n+3}E_{n+2} - E_{n+3}E_{n+1} - 7$.

(c) $\sum_{k=0}^{n} E_{k+2}E_k = E_{n+3}E_{n+2}$.

From the last Proposition, we have the following Corollary which gives sum formulas of Padovan-Perrin numbers (take $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

Corollary 4.11. *For $n \geq 0$, Padovan-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} S_k^2 = -S_{n+3}^2 - S_{n+2}^2 - 2S_{n+1}^2 + 2S_{n+3}S_{n+1} + 2S_{n+3}S_{n+2} + 1$.

(b) $\sum_{k=0}^{n} S_{k+1}S_k = S_{n+3}^2 + S_{n+2}^2 + S_{n+1}^2 - S_{n+3}S_{n+2} - S_{n+3}S_{n+1} - 1$.

(c) $\sum_{k=0}^{n} S_{k+2} S_k = S_{n+3} S_{n+2}.$

Taking $r = 0, s = 2, t = 1$ in Theorem 3.1, we obtain the following theorem.

Theorem 4.12. *If $r = 0, s = 2, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{2}((2n + 11) W_{n+3}^2 + (2n + 9) W_{n+2}^2 + (2n + 11) W_{n+1}^2 - 4(n + 5) W_{n+3} W_{n+2} - 4(n + 6) W_{n+3} W_{n+1} + 4(n + 6) W_{n+2} W_{n+1} - 9W_2^2 - 7W_1^2 - 9W_0^2 + 16W_2 W_1 + 20W_2 W_0 - 20W_1 W_0).$

(b) $\sum_{k=0}^{n} W_{k+1} W_k = \frac{1}{2}(-2(n + 5) W_{n+3}^2 - 2(n + 4) W_{n+2}^2 - 2(n + 6) W_{n+1}^2 + (4n + 19) W_{n+3} W_{n+2} + (4n + 23) W_{n+3} W_{n+1} - (4n + 23) W_{n+2} W_{n+1} + 8W_2^2 + 6W_1^2 + 10W_0^2 - 15W_2 W_1 - 19W_2 W_0 + 19W_1 W_0).$

(c) $\sum_{k=0}^{n} W_{k+2} W_k = \frac{1}{2}(2(n + 5) W_{n+3}^2 + 2(n + 4) W_{n+2}^2 + 2(n + 6) W_{n+1}^2 - (4n + 17) W_{n+3} W_{n+2} - (4n + 23) W_{n+3} W_{n+1} + (4n + 21) W_{n+2} W_{n+1} - 8W_2^2 - 6W_1^2 - 10W_0^2 + 13W_2 W_1 + 19W_2 W_0 - 17W_1 W_0).$

Proof.

(a) We use Theorem 3.1 (a). If we set $r = 0, s = 2, t = 1$ in Theorem 3.1 (a) then we have

$$\sum_{k=0}^{n} W_k^2 = \frac{g_1(x)}{-(x - 1)(x + 1)(-x + x^2 - 1)(-3x + x^2 + 1)}$$

where

$$
\begin{aligned}
g_1(x) =\ & (-(x^3 + 2x - 1)x^{n+3} W_{n+3}^2 - (x^3 + 2x - 1)x^{n+2} W_{n+2}^2 - (4x^5 - 7x^3 + 4x^2 + 2x - 1)x^{n+1} W_{n+1}^2 \\
& + 4x^{n+5} W_{n+3} W_{n+2} + 4x^{n+6} W_{n+3} W_{n+1} - 4(2x - 1)x^{n+4} W_{n+2} W_{n+1} + x^2(x^3 + 2x - 1)W_2^2 \\
& + x(x^3 + 2x - 1)W_1^2 + (4x^5 - 7x^3 + 4x^2 + 2x - 1)W_0^2 - 4x^4 W_2 W_1 - 4x^5 W_2 W_0 + 4(2x - 1)x^3 W_1 W_0)
\end{aligned}
$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$
\begin{aligned}
\sum_{k=0}^{n} W_k^2 =\ & \left. \frac{\frac{d}{dx}(g_1(x))}{\frac{d}{dx}(-(x - 1)(x + 1)(-x + x^2 - 1)(-3x + x^2 + 1))} \right|_{x=1} \\
=\ & \frac{1}{2}((2n + 11) W_{n+3}^2 + (2n + 9) W_{n+2}^2 + (2n + 11) W_{n+1}^2 - 4(n + 5) W_{n+3} W_{n+2} \\
& -4(n + 6) W_{n+3} W_{n+1} + 4(n + 6) W_{n+2} W_{n+1} - 9W_2^2 - 7W_1^2 - 9W_0^2 + 16W_2 W_1 \\
& + 20W_2 W_0 - 20W_1 W_0).
\end{aligned}
$$

(b) We use Theorem 3.1 (b). If we set $r = 0, s = 2, t = 1$ in Theorem 3.1 (b) then we have

$$\sum_{k=0}^{n} W_{k+1} W_k = \frac{g_2(x)}{-(x - 1)(x + 1)(-x + x^2 - 1)(-3x + x^2 + 1)}$$

where

$$
\begin{aligned}
g_2(x) =\ & (2x^2 x^{n+3} W_{n+3}^2 + 2x^{n+4} W_{n+2}^2 + 2x^2 x^{n+4} W_{n+1}^2 - (x^3 + 4x^2 - 1)x^{n+2} W_{n+3} W_{n+2} \\
& -(x^3 + 4x^2 - 1)x^{n+3} W_{n+3} W_{n+1} + (2x^4 + 7x^3 - 4x^2 - 2x + 1)x^{n+1} W_{n+2} W_{n+1} \\
& -2x^4 W_2^2 - 2x^3 W_1^2 - 2x^5 W_0^2 + x(x^3 + 4x^2 - 1)W_2 W_1 + (x^3 + 4x^2 - 1)x^2 W_2 W_0 \\
& -(2x^4 + 7x^3 - 4x^2 - 2x + 1)W_1 W_0)
\end{aligned}
$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\sum_{k=0}^{n} W_{k+1}W_k = \left. \frac{\frac{d}{dx}(g_2(x))}{\frac{d}{dx}(-(x-1)(x+1)(-x+x^2-1)(-3x+x^2+1))} \right|_{x=1}$$

$$= \frac{1}{2}(-2(n+5)W_{n+3}^2 - 2(n+4)W_{n+2}^2 - 2(n+6)W_{n+1}^2 + (4n+19)W_{n+3}W_{n+2}$$

$$+ (4n+23)W_{n+3}W_{n+1} - (4n+23)W_{n+2}W_{n+1} + 8W_2^2 + 6W_1^2 + 10W_0^2$$

$$-15W_2W_1 - 19W_2W_0 + 19W_1W_0).$$

(c) We use Theorem 3.1 (c). If we set $r = 0, s = 2, t = 1$ in Theorem 3.1 (c) then we have

$$\sum_{k=0}^{n} W_{k+2}W_k = \frac{g_3(x)}{-(x-1)(x+1)(-x+x^2-1)(-3x+x^2+1)}$$

where

$$g_3(x) = (-(4x-2)x^{n+3}W_{n+3}^2 - x^{n+2}(4x-2)W_{n+2}^2 - x^{n+4}(4x-2)W_{n+1}^2 + (-x^3+4x^2+1)x^{n+3}W_{n+3}W_{n+2}$$

$$+ (2x^4+7x^3-4x^2-2x+1)x^{n+1}W_{n+3}W_{n+1} - (2x-1)\left(-x^3+4x^2+1\right)x^{n+2}W_{n+2}W_{n+1}$$

$$+ (4x-2)x^2W_2^2 + x(4x-2)W_1^2 + (4x-2)x^3W_0^2 - (-x^3+4x^2+1)x^2W_2W_1$$

$$- (2x^4+7x^3-4x^2-2x+1)W_2W_0 + x(2x-1)(-x^3+4x^2+1)W_1W_0)$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\sum_{k=0}^{n} W_{k+2}W_k = \left. \frac{\frac{d}{dx}(g_3(x))}{\frac{d}{dx}(-(x-1)(x+1)(-x+x^2-1)(-3x+x^2+1))} \right|_{x=1}$$

$$= \frac{1}{2}(2(n+5)W_{n+3}^2 + 2(n+4)W_{n+2}^2 + 2(n+6)W_{n+1}^2 - (4n+17)W_{n+3}W_{n+2}$$

$$- (4n+23)W_{n+3}W_{n+1} + (4n+21)W_{n+2}W_{n+1} - 8W_2^2 - 6W_1^2 - 10W_0^2$$

$$+ 13W_2W_1 + 19W_2W_0 - 17W_1W_0).$$

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $Q_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 4.13. *For $n \geq 0$, Pell-Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} R_k^2 = \frac{1}{2}((2n+11)R_{n+3}^2 + (2n+9)R_{n+2}^2 + (2n+11)R_{n+1}^2 - 4(n+5)R_{n+3}R_{n+2} - 4(n+6)R_{n+3}R_{n+1} + 4(n+6)R_{n+2}R_{n+1} - 9).$

(b) $\sum_{k=0}^{n} R_{k+1}R_k = \frac{1}{2}(-2(n+5)R_{n+3}^2 - 2(n+4)R_{n+2}^2 - 2(n+6)R_{n+1}^2 + (4n+19)R_{n+3}R_{n+2} + (4n+23)R_{n+3}R_{n+1} - (4n+23)R_{n+2}R_{n+1} + 9).$

(c) $\sum_{k=0}^{n} R_{k+2}R_k = \frac{1}{2}(2(n+5)R_{n+3}^2 + 2(n+4)R_{n+2}^2 + 2(n+6)R_{n+1}^2 - (4n+17)R_{n+3}R_{n+2} - (4n+23)R_{n+3}R_{n+1} + (4n+21)R_{n+2}R_{n+1} - 9).$

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 4.14. *For $n \geq 0$, Pell-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} C_k^2 = \frac{1}{2}((2n+11)C_{n+3}^2 + (2n+9)C_{n+2}^2 + (2n+11)C_{n+1}^2 - 4(n+5)C_{n+3}C_{n+2} - 4(n+6)C_{n+3}C_{n+1} + 4(n+6)C_{n+2}C_{n+1} + 3).$

(b) $\sum_{k=0}^{n} C_{k+1}C_k = \frac{1}{2}(-2(n+5)C_{n+3}^2 - 2(n+4)C_{n+2}^2 - 2(n+6)C_{n+1}^2 + (4n+19)C_{n+3}C_{n+2} + (4n+23)C_{n+3}C_{n+1} - (4n+23)C_{n+2}C_{n+1} + 8).$

(c) $\sum_{k=0}^{n} C_{k+2}C_k = \frac{1}{2}(2(n+5)C_{n+3}^2 + 2(n+4)C_{n+2}^2 + 2(n+6)C_{n+1}^2 - (4n+17)C_{n+3}C_{n+2} - (4n+23)C_{n+3}C_{n+1} + (4n+21)C_{n+2}C_{n+1} - 8).$

Taking $r = 0, s = 1, t = 2$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.15. *If $r = 0, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{2}(W_{n+3}^2 + W_{n+2}^2 + 2W_{n+1}^2 - W_{n+3}W_{n+2} - 2W_{n+3}W_{n+1} - W_2^2 - W_1^2 - 2W_0^2 + W_2W_1 + 2W_2W_0).$

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{4}(-W_{n+3}^2 - W_{n+2}^2 - 4W_{n+1}^2 + 2W_{n+3}W_{n+2} + 4W_{n+3}W_{n+1} + W_2^2 + W_1^2 + 4W_0^2 - 2W_2W_1 - 4W_2W_0).$

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{2}(W_{n+3}W_{n+2} - W_1W_2).$

From the last Proposition, we have the following Corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

Corollary 4.16. *For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} Q_k^2 = \frac{1}{2}(Q_{n+3}^2 + Q_{n+2}^2 + 2Q_{n+1}^2 - Q_{n+3}Q_{n+2} - 2Q_{n+3}Q_{n+1} - 1).$

(b) $\sum_{k=0}^{n} Q_{k+1}Q_k = \frac{1}{4}(-Q_{n+3}^2 - Q_{n+2}^2 - 4Q_{n+1}^2 + 2Q_{n+3}Q_{n+2} + 4Q_{n+3}Q_{n+1}).$

(c) $\sum_{k=0}^{n} Q_{k+2}Q_k = \frac{1}{2}(Q_{n+3}Q_{n+2} - 1).$

Taking $W_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$ in the last Proposition, we have the following Corollary which presents sum formulas of Jacobsthal-Perrin numbers.

Corollary 4.17. *For $n \geq 0$, Jacobsthal-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} L_k^2 = \frac{1}{2}(L_{n+3}^2 + L_{n+2}^2 + 2L_{n+1}^2 - L_{n+3}L_{n+2} - 2L_{n+3}L_{n+1} - 10).$

(b) $\sum_{k=0}^{n} L_{k+1}L_k = \frac{1}{4}(-L_{n+3}^2 - L_{n+2}^2 - 4L_{n+1}^2 + 2L_{n+3}L_{n+2} + 4L_{n+3}L_{n+1} + 16).$

(c) $\sum_{k=0}^{n} L_{k+2}L_k = \frac{1}{2}L_{n+3}L_{n+2}.$

Taking $r = 1, s = 0, t = 1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 4.18. *If $r = 1, s = 0, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{3}(-W_{n+3}^2 - 4W_{n+2}^2 - 4W_{n+1}^2 + 4W_{n+3}W_{n+2} + 2W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1} + W_2^2 + 4W_1^2 + 4W_0^2 - 4W_2W_1 - 2W_2W_0 - 2W_1W_0).$

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{3}(W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - W_{n+3}W_{n+2} + W_{n+3}W_{n+1} - 2W_{n+2}W_{n+1} - W_2^2 - W_1^2 - W_0^2 + W_2W_1 - W_2W_0 + 2W_1W_0).$

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{3}(2W_{n+3}^2 + 2W_{n+2}^2 + 2W_{n+1}^2 - 2W_{n+3}W_{n+2} - W_{n+3}W_{n+1} - W_{n+2}W_{n+1} - 2W_2^2 - 2W_1^2 - 2W_0^2 + 2W_2W_1 + W_2W_0 + W_1W_0).$

From the last proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 4.19. *For $n \geq 0$, Narayana numbers have the following properties:*

(a) $\sum_{k=0}^{n} N_k^2 = \frac{1}{3}(-N_{n+3}^2 - 4N_{n+2}^2 - 4N_{n+1}^2 + 4N_{n+3}N_{n+2} + 2N_{n+3}N_{n+1} + 2N_{n+2}N_{n+1} + 1).$

(b) $\sum_{k=0}^{n} N_{k+1}N_k = \frac{1}{3}(N_{n+3}^2 + N_{n+2}^2 + N_{n+1}^2 - N_{n+3}N_{n+2} + N_{n+3}N_{n+1} - 2N_{n+2}N_{n+1} - 1).$

(c) $\sum_{k=0}^{n} N_{k+2}N_k = \frac{1}{3}(2N_{n+3}^2 + 2N_{n+2}^2 + 2N_{n+1}^2 - 2N_{n+3}N_{n+2} - N_{n+3}N_{n+1} - N_{n+2}N_{n+1} - 2).$

Taking $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

Corollary 4.20. *For $n \geq 0$, Narayana-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} U_k^2 = \frac{1}{3}(-U_{n+3}^2 - 4U_{n+2}^2 - 4U_{n+1}^2 + 4U_{n+3}U_{n+2} + 2U_{n+3}U_{n+1} + 2U_{n+2}U_{n+1} + 25)$.

(b) $\sum_{k=0}^{n} U_{k+1}U_k = \frac{1}{3}(U_{n+3}^2 + U_{n+2}^2 + U_{n+1}^2 - U_{n+3}U_{n+2} + U_{n+3}U_{n+1} - 2U_{n+2}U_{n+1} - 7)$.

(c) $\sum_{k=0}^{n} U_{k+2}U_k = \frac{1}{3}(2U_{n+3}^2 + 2U_{n+2}^2 + 2U_{n+1}^2 - 2U_{n+3}U_{n+2} - U_{n+3}U_{n+1} - U_{n+2}U_{n+1} - 14)$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $W_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

Corollary 4.21. *For $n \geq 0$, Narayana-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} H_k^2 = \frac{1}{3}(-H_{n+3}^2 - 4H_{n+2}^2 - 4H_{n+1}^2 + 4H_{n+3}H_{n+2} + 2H_{n+3}H_{n+1} + 2H_{n+2}H_{n+1} + 28)$.

(b) $\sum_{k=0}^{n} H_{k+1}H_k = \frac{1}{3}(H_{n+3}^2 + H_{n+2}^2 + H_{n+1}^2 - H_{n+3}H_{n+2} + H_{n+3}H_{n+1} - 2H_{n+2}H_{n+1} - 19)$.

(c) $\sum_{k=0}^{n} H_{k+2}H_k = \frac{1}{3}(2H_{n+3}^2 + 2H_{n+2}^2 + 2H_{n+1}^2 - 2H_{n+3}H_{n+2} - H_{n+3}H_{n+1} - H_{n+2}H_{n+1} - 20)$.

Taking $r = 1, s = 1, t = 2$ in Theorem 3.1, we obtain the following theorem.

Theorem 4.22. *If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{63}((6n + 35)\,W_{n+3}^2 + (18n + 90)\,W_{n+2}^2 + (24n + 101)\,W_{n+1}^2 - 6\,(3n + 16)\,W_{n+3}W_{n+2} - 4\,(3n + 16)\,W_{n+3}W_{n+1} + 12W_{n+2}W_{n+1} - 29W_2^2 - 72W_1^2 - 77W_0^2 + 78W_2W_1 + 52W_2W_0 - 12W_1W_0)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{63}(-(3n + 13)\,W_{n+3}^2 - 3\,(3n + 14)\,W_{n+2}^2 - 4\,(3n + 16)\,W_{n+1}^2 + (9n + 45)\,W_{n+3}W_{n+2} + 2\,(3n + 22)\,W_{n+3}W_{n+1} - 27W_{n+2}W_{n+1} + 10W_2^2 + 33W_1^2 + 52W_0^2 - 36W_2W_1 - 38W_2W_0 + 27W_1W_0)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{63}(-(3n + 10)\,W_{n+3}^2 - (9n + 54)\,W_{n+2}^2 - 4\,(3n + 13)\,W_{n+1}^2 + (9n + 57)\,W_{n+3}W_{n+2} + (6n + 17)\,W_{n+3}W_{n+1} - 6W_{n+2}W_{n+1} + 7W_2^2 + 45W_1^2 + 40W_0^2 - 48W_2W_1 - 11W_2W_0 + 6W_1W_0)$.

Proof.

(a) We use Theorem 3.1 (a). If we set $r = 1, s = 1, t = 2$ in Theorem 3.1 (a) then we have

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{g_4(x)}{-(x - 1)(4x - 1)(x + x^2 + 1)(2x + 4x^2 + 1)}$$

where

$$\begin{aligned}
g_4(x) =\ & -x^{n+3}(4x^3 + 2x^2 + x - 1)W_{n+3}^2 - (4x^4 + 10x^3 + 3x^2 + 2x - 1)x^{n+2}W_{n+2}^2 \\
& -(4x^5 + 2x^4 + 13x^3 + 4x^2 + 2x - 1)x^{n+1}W_{n+1}^2 + 2(2x + 1)^2 x^{n+4}W_{n+3}W_{n+2} \\
& +4(2x^2 + 1)x^{n+4}W_{n+3}W_{n+1} - 4(x - 1)(2x + 1)x^{n+4}W_{n+2}W_{n+1} \\
& +x^2(4x^3 + 2x^2 + x - 1)W_2^2 + x(4x^4 + 10x^3 + 3x^2 + 2x - 1)W_1^2 \\
& +(4x^5 + 2x^4 + 13x^3 + 4x^2 + 2x - 1)W_0^2 - 2(2x + 1)^2 x^3 W_2 W_1 \\
& -4(2x^2 + 1)x^3 W_2 W_0 + 4(2x + 1)(x - 1)x^3 W_1 W_0.
\end{aligned}$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}
\sum_{k=0}^{n} W_k^2 =\ & \left. \frac{\frac{d}{dx}(g_4(x))}{\frac{d}{dx}(-(x - 1)(4x - 1)(x + x^2 + 1)(2x + 4x^2 + 1))} \right|_{x=1} \\
=\ & \frac{1}{63}((6n + 35)\,W_{n+3}^2 + (18n + 90)\,W_{n+2}^2 + (24n + 101)\,W_{n+1}^2 - 6\,(3n + 16)\,W_{n+3}W_{n+2} \\
& -4\,(3n + 16)\,W_{n+3}W_{n+1} + 12W_{n+2}W_{n+1} - 29W_2^2 - 72W_1^2 - 77W_0^2 + 78W_2 W_1 \\
& +52W_2 W_0 - 12W_1 W_0).
\end{aligned}$$

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(b) We use Theorem 3.1 (b). If we set $r = 1, s = 1, t = 2$ in Theorem 3.1 (b) then we have

$$\sum_{k=0}^{n} x^k W_{k+1} W_k = \frac{g_5(x)}{-(x-1)(4x-1)(x+x^2+1)(2x+4x^2+1)}$$

where

$$
\begin{aligned}
g_5(x) \quad = \quad & (2x^2+1)x^{n+3}W_{n+3}^2 + 3(2x+1)x^{n+4}W_{n+2}^2 + 4(2x^2+1)x^{n+4}W_{n+1}^2 \\
& -(8x^3+x^2+x-1)x^{n+2}W_{n+3}W_{n+2} + 2(-4x^3-x^2+x+1)x^{n+3}W_{n+3}W_{n+1} \\
& -(-8x^5-6x^4+9x^3+4x^2+2x-1)x^{n+1}W_{n+2}W_{n+1} - (2x^2+1)x^2W_2^2 \\
& -3(2x+1)x^3W_1^2 - 4(2x^2+1)x^3W_0^2 + x(8x^3+x^2+x-1)W_2W_1 \\
& -2(-4x^3-x^2+x+1)x^2W_2W_0 + (-8x^5-6x^4+9x^3+4x^2+2x-1)W_1W_0.
\end{aligned}
$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$
\begin{aligned}
\sum_{k=0}^{n} W_{k+1} W_k \quad = \quad & \left. \frac{\frac{d}{dx}(g_5(x))}{\frac{d}{dx}(-(x-1)(4x-1)(x+x^2+1)(2x+4x^2+1))} \right|_{x=1} \\
= \quad & \frac{1}{63} \big(-(3n+13)W_{n+3}^2 - 3(3n+14)W_{n+2}^2 - 4(3n+16)W_{n+1}^2 \\
& + (9n+45)W_{n+3}W_{n+2} + 2(3n+22)W_{n+3}W_{n+1} - 27W_{n+2}W_{n+1} \\
& + 10W_2^2 + 33W_1^2 + 52W_0^2 - 36W_2W_1 - 38W_2W_0 + 27W_1W_0 \big)
\end{aligned}
$$

(c) We use Theorem 3.1 (c). If we set $r = 1, s = 1, t = 2$ in Theorem 3.1 (c) then we have

$$\sum_{k=0}^{n} x^k W_{k+2} W_k = \frac{g_6(x)}{-(x-1)(4x-1)(x+x^2+1)(2x+4x^2+1)}$$

where

$$
\begin{aligned}
g_6(x) \quad = \quad & (x+2)x^{n+3}W_{n+3}^2 + (8x^4+2x^3-2x+1)x^{n+2}W_{n+2}^2 + 4(x+2)x^{n+4}W_{n+1}^2 \\
& -(2x+1)(4x^3-x^2+x-1)x^{n+2}W_{n+3}W_{n+2} - (-4x^4+7x^3+2x^2+2x-1)x^{n+1}W_{n+3}W_{n+1} \\
& +2(x-1)(4x^3-x^2+x-1)x^{n+2}W_{n+2}W_{n+1} - (x+2)x^2W_2^2 - x(8x^4+2x^3-2x+1)W_1^2 \\
& -4(x+2)x^3W_0^2 + x(2x+1)(4x^3-x^2+x-1)W_2W_1 \\
& +(-4x^4+7x^3+2x^2+2x-1)W_2W_0 - 2x(x-1)(4x^3-x^2+x-1)W_1W_0.
\end{aligned}
$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$
\begin{aligned}
\sum_{k=0}^{n} W_{k+2} W_k \quad = \quad & \left. \frac{\frac{d}{dx}(g_6(x))}{\frac{d}{dx}(-(x-1)(4x-1)(x+x^2+1)(2x+4x^2+1))} \right|_{x=1} \\
= \quad & \frac{1}{63} \big(-(3n+10)W_{n+3}^2 - (9n+54)W_{n+2}^2 - 4(3n+13)W_{n+1}^2 \\
& + (9n+57)W_{n+3}W_{n+2} + (6n+17)W_{n+3}W_{n+1} - 6W_{n+2}W_{n+1} \\
& + 7W_2^2 + 45W_1^2 + 40W_0^2 - 48W_2W_1 - 11W_2W_0 + 6W_1W_0 \big).
\end{aligned}
$$

From the last theorem, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

Corollary 4.23. *For $n \geq 0$, third order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} J_k^2 = \frac{1}{63}((6n+35)J_{n+3}^2 + (18n+90)J_{n+2}^2 + (24n+101)J_{n+1}^2 - 6(3n+16)J_{n+3}J_{n+2} - 4(3n+16)J_{n+3}J_{n+1} + 12J_{n+2}J_{n+1} - 23)$.

(b) $\sum_{k=0}^{n} J_{k+1}J_k = \frac{1}{63}(-(3n+13)J_{n+3}^2 - 3(3n+14)J_{n+2}^2 - 4(3n+16)J_{n+1}^2 + (9n+45)J_{n+3}J_{n+2} + 2(3n+22)J_{n+3}J_{n+1} - 27J_{n+2}J_{n+1} + 7)$.

(c) $\sum_{k=0}^{n} J_{k+2}J_k = \frac{1}{63}(-(3n+10)J_{n+3}^2 - (9n+54)J_{n+2}^2 - 4(3n+13)J_{n+1}^2 + (9n+57)J_{n+3}J_{n+2} + (6n+17)J_{n+3}J_{n+1} - 6J_{n+2}J_{n+1} + 4)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$ in the last theorem, we have the following corollary which presents sum formulas of third order Jacobsthal-Lucas numbers.

Corollary 4.24. *For $n \geq 0$, third order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} j_k^2 = \frac{1}{63}((6n+35)j_{n+3}^2 + (18n+90)j_{n+2}^2 + (24n+101)j_{n+1}^2 - 6(3n+16)j_{n+3}j_{n+2} - 4(3n+16)j_{n+3}j_{n+1} + 12j_{n+2}j_{n+1} - 219)$.

(b) $\sum_{k=0}^{n} j_{k+1}j_k = \frac{1}{63}(-(3n+13)j_{n+3}^2 - 3(3n+14)j_{n+2}^2 - 4(3n+16)j_{n+1}^2 + (9n+45)j_{n+3}j_{n+2} + 2(3n+22)j_{n+3}j_{n+1} - 27j_{n+2}j_{n+1} - 15)$.

(c) $\sum_{k=0}^{n} j_{k+2}j_k = \frac{1}{63}(-(3n+10)j_{n+3}^2 - (9n+54)j_{n+2}^2 - 4(3n+13)j_{n+1}^2 + (9n+57)j_{n+3}j_{n+2} + (6n+17)j_{n+3}j_{n+1} - 6j_{n+2}j_{n+1} + 42)$.

Taking $r = 2, s = 3, t = 5$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.25. *If $r = 2, s = 3, t = 5$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{495}(37W_{n+3}^2 + 253W_{n+2}^2 + 430W_{n+1}^2 - 182W_{n+3}W_{n+2} - 170W_{n+3}W_{n+1} + 260W_{n+2}W_{n+1} - 37W_2^2 - 253W_1^2 - 430W_0^2 + 182W_2W_1 + 170W_2W_0 - 260W_1W_0)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{495}(-17W_{n+3}^2 - 143W_{n+2}^2 - 425W_{n+1}^2 + 97W_{n+3}W_{n+2} + 145W_{n+3}W_{n+1} - 280W_{n+2}W_{n+1} + 17W_2^2 + 143W_1^2 + 425W_0^2 - 97W_2W_1 - 145W_2W_0 + 280W_1W_0)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{495}(-8W_{n+3}^2 - 242W_{n+2}^2 - 200W_{n+1}^2 + 133W_{n+3}W_{n+2} + 10W_{n+3}W_{n+1} - 190W_{n+2}W_{n+1} + 8W_2^2 + 242W_1^2 + 200W_0^2 - 133W_2W_1 - 10W_2W_0 + 190W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of 3-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 4.26. *For $n \geq 0$, 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} G_k^2 = \frac{1}{495}(37G_{n+3}^2 + 253G_{n+2}^2 + 430G_{n+1}^2 - 182G_{n+3}G_{n+2} - 170G_{n+3}G_{n+1} + 260G_{n+2}G_{n+1} - 37)$.

(b) $\sum_{k=0}^{n} G_{k+1}G_k = \frac{1}{495}(-17G_{n+3}^2 - 143G_{n+2}^2 - 425G_{n+1}^2 + 97G_{n+3}G_{n+2} + 145G_{n+3}G_{n+1} - 280G_{n+2}G_{n+1} + 17)$.

(c) $\sum_{k=0}^{n} G_{k+2}G_k = \frac{1}{495}(-8G_{n+3}^2 - 242G_{n+2}^2 - 200G_{n+1}^2 + 133G_{n+3}G_{n+2} + 10G_{n+3}G_{n+1} - 190G_{n+2}G_{n+1} + 8)$.

Taking $G_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

Corollary 4.27. *For $n \geq 0$, Lucas 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} H_k^2 = \frac{1}{495}(37H_{n+3}^2 + 253H_{n+2}^2 + 430H_{n+1}^2 - 182H_{n+3}H_{n+2} - 170H_{n+3}H_{n+1} + 260H_{n+2}H_{n+1} - 1402)$.

(b) $\sum_{k=0}^{n} H_{k+1}H_k = \frac{1}{495}(-17H_{n+3}^2 - 143H_{n+2}^2 - 425H_{n+1}^2 + 97H_{n+3}H_{n+2} + 145H_{n+3}H_{n+1} - 280H_{n+2}H_{n+1} + 1487)$.

(c) $\sum_{k=0}^{n} H_{k+2}H_k = \frac{1}{495}(-8H_{n+3}^2 - 242H_{n+2}^2 - 200H_{n+1}^2 + 133H_{n+3}H_{n+2} + 10H_{n+3}H_{n+1} - 190H_{n+2}H_{n+1} + 1748)$.

From the last proposition, we have the following corollary which gives sum formulas of modified 3-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 4.28. *For $n \geq 0$, modified 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} E_k^2 = \frac{1}{495}(37E_{n+3}^2 + 253E_{n+2}^2 + 430E_{n+1}^2 - 182E_{n+3}E_{n+2} - 170E_{n+3}E_{n+1} + 260E_{n+2}E_{n+1} - 108)$.

(b) $\sum_{k=0}^{n} E_{k+1}E_k = \frac{1}{495}(-17E_{n+3}^2 - 143E_{n+2}^2 - 425E_{n+1}^2 + 97E_{n+3}E_{n+2} + 145E_{n+3}E_{n+1} - 280E_{n+2}E_{n+1} + 63)$.

(c) $\sum_{k=0}^{n} E_{k+2}E_k = \frac{1}{495}(-8E_{n+3}^2 - 242E_{n+2}^2 - 200E_{n+1}^2 + 133E_{n+3}E_{n+2} + 10E_{n+3}E_{n+1} - 190E_{n+2}E_{n+1} + 117)$.

4.2 The Case $x = -1$

In this subsection we consider the special case $x = -1$.

In this section, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}(-1)^k W_k^2, \sum_{k=0}^{n}(-1)^k W_{k+2} W_k$ and $\sum_{k=0}^{n}(-1)^k W_{k+1} W_k$ for the specific case of the sequence $\{W_n\}$.

Taking $r = s = t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.29. *If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{4}((-1)^n (W_{n+3}^2 - 2W_{n+2}^2 + 3W_{n+1}^2 - 2W_{n+1}W_{n+3}) + W_2^2 - 2W_1^2 + 3W_0^2 - 2W_0 W_2)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1} W_k = \frac{1}{4}((-1)^n (W_{n+3}^2 - W_{n+1}^2 - 2W_{n+3}W_{n+2} + 2W_{n+2}W_{n+1}) + W_2^2 - W_0^2 - 2W_1 W_2 + 2W_1 W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2} W_k = \frac{1}{4}((-1)^n (W_{n+3}^2 - 2W_{n+2}^2 - W_{n+1}^2 + 2W_{n+3}W_{n+1} - 4W_{n+2}W_{n+1}) + W_2^2 - 2W_1^2 - W_0^2 + 2W_2 W_0 - 4W_1 W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 4.30. *For $n \geq 0$, Tribonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k T_k^2 = \frac{1}{4}((-1)^n (T_{n+3}^2 - 2T_{n+2}^2 + 3T_{n+1}^2 - 2T_{n+1}T_{n+3}) - 1)$.

(b) $\sum_{k=0}^{n}(-1)^k T_{k+1} T_k = \frac{1}{4}((-1)^n (T_{n+3}^2 - T_{n+1}^2 - 2T_{n+3}T_{n+2} + 2T_{n+2}T_{n+1}) - 1)$.

(c) $\sum_{k=0}^{n}(-1)^k T_{k+2} T_k = \frac{1}{4}((-1)^n (T_{n+3}^2 - 2T_{n+2}^2 - T_{n+1}^2 + 2T_{n+3}T_{n+1} - 4T_{n+2}T_{n+1}) - 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 4.31. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k K_k^2 = \frac{1}{4}((-1)^n (K_{n+3}^2 - 2K_{n+2}^2 + 3K_{n+1}^2 - 2K_{n+1}K_{n+3}) + 16)$.

(b) $\sum_{k=0}^{n}(-1)^k K_{k+1} K_k = \frac{1}{4}(-1)^n (K_{n+3}^2 - K_{n+1}^2 - 2K_{n+3}K_{n+2} + 2K_{n+2}K_{n+1})$.

(c) $\sum_{k=0}^{n}(-1)^k K_{k+2} K_k = \frac{1}{4}((-1)^n (K_{n+3}^2 - 2K_{n+2}^2 - K_{n+1}^2 + 2K_{n+3}K_{n+1} - 4K_{n+2}K_{n+1}) + 4)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.32. *If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{15}((-1)^n (W_{n+3}^2 - 9W_{n+2}^2 + 14W_{n+1}^2 + 2W_{n+3}W_{n+2} + 4W_{n+2}W_{n+1} - 6W_{n+3}W_{n+1}) + W_2^2 - 9W_1^2 + 14W_0^2 + 2W_2 W_1 - 6W_2 W_0 + 4W_1 W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1} W_k = \frac{1}{5}((-1)^n (W_{n+3}^2 + W_{n+2}^2 - W_{n+1}^2 - 3W_{n+3}W_{n+2} - W_{n+3}W_{n+1} + 4W_{n+2}W_{n+1}) + W_2^2 + W_1^2 - W_0^2 - 3W_2 W_1 - W_2 W_0 + 4W_1 W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2} W_k = \frac{1}{15}((-1)^n (4W_{n+3}^2 - 6W_{n+2}^2 - 4W_{n+1}^2 - 7W_{n+3}W_{n+2} + 6W_{n+3}W_{n+1} - 14 W_{n+2}W_{n+1}) + 4W_2^2 - 6W_1^2 - 4W_0^2 - 7W_2 W_1 + 6W_2 W_0 - 14W_1 W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1$).

Corollary 4.33. *For $n \geq 0$, third-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k P_k^2 = \frac{1}{15}((-1)^n (P_{n+3}^2-9P_{n+2}^2+14P_{n+1}^2+2P_{n+3}P_{n+2}+4P_{n+2}P_{n+1}-6P_{n+3}P_{n+1})-1)$.

(b) $\sum_{k=0}^{n}(-1)^k P_{k+1}P_k = \frac{1}{15}((-1)^n (P_{n+3}^2-9P_{n+2}^2+14P_{n+1}^2+2P_{n+3}P_{n+2}+4P_{n+2}P_{n+1}-6P_{n+3}P_{n+1})-1)$.

(c) $\sum_{k=0}^{n}(-1)^k P_{k+2}P_k = \frac{1}{15}((-1)^n (4P_{n+3}^2 - 6P_{n+2}^2 - 4P_{n+1}^2 - 7P_{n+3}P_{n+2} + 6P_{n+3}P_{n+1} - 14 P_{n+2}P_{n+1}) - 4)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Lucas numbers.

Corollary 4.34. *For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k Q_k^2 = \frac{1}{15}((-1)^n (Q_{n+3}^2-9Q_{n+2}^2+14Q_{n+1}^2+2Q_{n+3}Q_{n+2}+4Q_{n+2}Q_{n+1}-6Q_{n+3}Q_{n+1})+66)$.

(b) $\sum_{k=0}^{n}(-1)^k Q_{k+1}Q_k = \frac{1}{5}((-1)^n (Q_{n+3}^2+Q_{n+2}^2-Q_{n+1}^2-3Q_{n+3}Q_{n+2}-Q_{n+3}Q_{n+1}+4Q_{n+2}Q_{n+1})+1)$.

(c) $\sum_{k=0}^{n}(-1)^k Q_{k+2}Q_k = \frac{1}{15}((-1)^n (4Q_{n+3}^2 - 6Q_{n+2}^2 - 4Q_{n+1}^2 - 7Q_{n+3}Q_{n+2} + 6Q_{n+3}Q_{n+1} - 14 Q_{n+2}Q_{n+1}) + 24)$.

From the last proposition, we have the following corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 4.35. *For $n \geq 0$, third-order modified Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k E_k^2 = \frac{1}{15}((-1)^n (E_{n+3}^2-9E_{n+2}^2+14E_{n+1}^2+2E_{n+3}E_{n+2}+4E_{n+2}E_{n+1}-6E_{n+3}E_{n+1})-6)$.

(b) $\sum_{k=0}^{n}(-1)^k E_{k+1}E_k = \frac{1}{5}((-1)^n (E_{n+3}^2+E_{n+2}^2-E_{n+1}^2-3E_{n+3}E_{n+2}-E_{n+3}E_{n+1}+4E_{n+2}E_{n+1})-1)$.

(c) $\sum_{k=0}^{n}(-1)^k E_{k+2}E_k = \frac{1}{15}((-1)^n (4E_{n+3}^2 - 6E_{n+2}^2 - 4E_{n+1}^2 - 7E_{n+3}E_{n+2} + 6E_{n+3}E_{n+1} - 14 E_{n+2}E_{n+1}) - 9)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.36. *If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{5}((-1)^n (3W_{n+3}^2-3W_{n+2}^2+2W_{n+1}^2+2W_{n+3}W_{n+2}-2W_{n+3}W_{n+1}-4W_{n+2}W_{n+1})+3W_2^2 - 3W_1^2 + 2W_0^2 + 2W_2W_1 - 2W_2W_0 - 4W_1W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{5}((-1)^n (W_{n+3}^2-W_{n+2}^2-W_{n+1}^2-W_{n+3}W_{n+2}+W_{n+3}W_{n+1}+2W_{n+2}W_{n+1})+W_2^2 - W_1^2 - W_0^2 - W_2W_1 + W_2W_0 + 2W_1W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{5}((-1)^n (2W_{n+3}^2 - 2W_{n+2}^2 - 2W_{n+1}^2 + 3W_{n+3}W_{n+2} + 2W_{n+3}W_{n+1} - 6W_{n+2}W_{n+1}) + 2W_2^2 - 2W_1^2 - 2W_0^2 + 3W_2W_1 - 6W_1W_0 + 2W_2W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

Corollary 4.37. *For $n \geq 0$, Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k P_k^2 = \frac{1}{5}((-1)^n (3P_{n+3}^2-3P_{n+2}^2+2P_{n+1}^2+2P_{n+3}P_{n+2}-2P_{n+3}P_{n+1}-4P_{n+2}P_{n+1})-2)$.

(b) $\sum_{k=0}^{n}(-1)^k P_{k+1}P_k = \frac{1}{5}((-1)^n (P_{n+3}^2-P_{n+2}^2-P_{n+1}^2-P_{n+3}P_{n+2}+P_{n+3}P_{n+1}+2P_{n+2}P_{n+1})+1)$.

(c) $\sum_{k=0}^{n}(-1)^k P_{k+2}P_k = \frac{1}{5}((-1)^n (2P_{n+3}^2-2P_{n+2}^2-2P_{n+1}^2+3P_{n+3}P_{n+2}+2P_{n+3}P_{n+1}-6P_{n+2}P_{n+1})-3)$.

Taking $W_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Perrin numbers.

Corollary 4.38. *For $n \geq 0$, Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k E_k^2 = \frac{1}{5}((-1)^n (3E_{n+3}^2 - 3E_{n+2}^2 + 2E_{n+1}^2 + 2E_{n+3}E_{n+2} - 2E_{n+3}E_{n+1} - 4E_{n+2}E_{n+1}) + 18)$.

(b) $\sum_{k=0}^{n}(-1)^k E_{k+1}E_k = \frac{1}{5}((-1)^n (E_{n+3}^2 - E_{n+2}^2 - E_{n+1}^2 - E_{n+3}E_{n+2} + E_{n+3}E_{n+1} + 2E_{n+2}E_{n+1}) + 1)$.

(c) $\sum_{k=0}^{n}(-1)^k E_{k+2}E_k = \frac{1}{5}((-1)^n (2E_{n+3}^2 - 2E_{n+2}^2 - 2E_{n+1}^2 + 3E_{n+3}E_{n+2} + 2E_{n+3}E_{n+1} - 6E_{n+2}E_{n+1}) + 2)$.

From the last proposition, we have the following corollary which gives sum formulas of Padovan-Perrin numbers (take $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

Corollary 4.39. *For $n \geq 0$, Padovan-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k S_k^2 = \frac{1}{5}((-1)^n (3S_{n+3}^2 - 3S_{n+2}^2 + 2S_{n+1}^2 + 2S_{n+3}S_{n+2} - 2S_{n+3}S_{n+1} - 4S_{n+2}S_{n+1}) + 3)$.

(b) $\sum_{k=0}^{n}(-1)^k S_{k+1}S_k = \frac{1}{5}((-1)^n (S_{n+3}^2 - S_{n+2}^2 - S_{n+1}^2 - S_{n+3}S_{n+2} + S_{n+3}S_{n+1} + 2S_{n+2}S_{n+1}) + 1)$.

(c) $\sum_{k=0}^{n}(-1)^k S_{k+2}S_k = \frac{1}{5}((-1)^n (2S_{n+3}^2 - 2S_{n+2}^2 - 2S_{n+1}^2 + 3S_{n+3}S_{n+2} + 2S_{n+3}S_{n+1} - 6S_{n+2}S_{n+1}) + 2)$.

Taking $r = 0, s = 2, t = 1$ in Theorem 3.1, we obtain the following theorem.

Theorem 4.40. *If $r = 0, s = 2, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{10}((-1)^n ((4n+17) W_{n+3}^2 - (4n+13) W_{n+2}^2 - (4n+11) W_{n+1}^2 + 4(n+5) W_{n+3}W_{n+2} - 4(n+6) W_{n+3}W_{n+1} - 4(3n+14) W_{n+2}W_{n+1}) + 13W_2^2 - 9W_1^2 - 7W_0^2 + 16W_2W_1 - 20W_2W_0 - 44W_1W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{10}((-1)^n (2(n+5) W_{n+3}^2 - 2(n+4) W_{n+2}^2 - 2(n+6) W_{n+1}^2 + (2n+9) W_{n+3}W_{n+2} - (2n+11) W_{n+3}W_{n+1} - (6n+25) W_{n+2}W_{n+1}) + 8W_2^2 - 6W_1^2 - 10W_0^2 + 7W_2W_1 - 9W_2W_0 - 19W_1W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{10}((-1)^n (2(3n+11) W_{n+3}^2 - 2(3n+8) W_{n+2}^2 - 2(3n+14) W_{n+1}^2 + (6n+29) W_{n+3}W_{n+2} - (6n+25) W_{n+3}W_{n+1} - (18n+81) W_{n+2}W_{n+1}) + 16W_2^2 - 10W_1^2 - 22W_0^2 + 23W_2W_1 - 19W_2W_0 - 63W_1W_0)$.

Proof. The proof can be given exactly as in Theorem 4.12, just take $x = -1$ after using L'Hospital rule.

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $Q_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 4.41. *For $n \geq 0$, Pell-Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k R_k^2 = \frac{1}{10}((-1)^n ((4n+17) R_{n+3}^2 - (4n+13) R_{n+2}^2 - (4n+11) R_{n+1}^2 + 4(n+5) R_{n+3}R_{n+2} - 4(n+6) R_{n+3}R_{n+1} - 4(3n+14) R_{n+2}R_{n+1}) - 51)$.

(b) $\sum_{k=0}^{n}(-1)^k R_{k+1}R_k = \frac{1}{10}((-1)^n (2(n+5) R_{n+3}^2 - 2(n+4) R_{n+2}^2 - 2(n+6) R_{n+1}^2 + (2n+9) R_{n+3}R_{n+2} - (2n+11) R_{n+3}R_{n+1} - (6n+25) R_{n+2}R_{n+1}) - 29)$.

(c) $\sum_{k=0}^{n}(-1)^k R_{k+2}R_k = \frac{1}{10}((-1)^n (2(3n+11) R_{n+3}^2 - 2(3n+8) R_{n+2}^2 - 2(3n+14) R_{n+1}^2 + (6n+29) R_{n+3}R_{n+2} - (6n+25) R_{n+3}R_{n+1} - (18n+81) R_{n+2}R_{n+1}) - 75)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 4.42. *For $n \geq 0$, Pell-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k C_k^2 = \frac{1}{10}((-1)^n ((4n+17) C_{n+3}^2 - (4n+13) C_{n+2}^2 - (4n+11) C_{n+1}^2 + 4(n+5) C_{n+3}C_{n+2} - 4(n+6) C_{n+3}C_{n+1} - 4(3n+14) C_{n+2}C_{n+1}) - 131)$.

(b) $\sum_{k=0}^{n}(-1)^k C_{k+1}C_k = \frac{1}{10}((-1)^n (2(n+5)C_{n+3}^2 - 2(n+4)C_{n+2}^2 - 2(n+6)C_{n+1}^2 + (2n+9)$
$C_{n+3}C_{n+2} - (2n+11)C_{n+3}C_{n+1} - (6n+25)C_{n+2}C_{n+1}) - 112).$

(c) $\sum_{k=0}^{n}(-1)^k C_{k+2}C_k = \frac{1}{10}((-1)^n (2(3n+11)C_{n+3}^2 - 2(3n+8)C_{n+2}^2 - 2(3n+14)C_{n+1}^2$
$+ (6n+29)C_{n+3}C_{n+2} - (6n+25)C_{n+3}C_{n+1} - (18n+81)C_{n+2}C_{n+1}) - 248).$

Taking $r = 0, s = 1, t = 2$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.43. *If $r = 0, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{16}((-1)^n (3W_{n+3}^2 - 3W_{n+2}^2 + 4W_{n+1}^2 + 2W_{n+3}W_{n+2} - 4W_{n+3}W_{n+1} - 4W_{n+2}W_{n+1}) +$
$3W_2^2 - 3W_1^2 + 4W_0^2 + 2W_2W_1 - 4W_2W_0 - 4W_1W_0).$

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{16}((-1)^n (W_{n+3}^2 - W_{n+2}^2 - 4W_{n+1}^2 - 2W_{n+3}W_{n+2} + 4W_{n+3}W_{n+1} +$
$4W_{n+2}W_{n+1}) - 2W_2W_1 + 4W_2W_0 + 4W_1W_0 + W_2^2 - W_1^2 - 4W_0^2).$

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{16}((-1)^n (W_{n+3}^2 - W_{n+2}^2 - 4W_{n+1}^2 + 6W_{n+3}W_{n+2} + 4W_{n+3}W_{n+1} - 12$
$W_{n+2}W_{n+1}) + W_2^2 - W_1^2 - 4W_0^2 + 6W_2W_1 + 4W_2W_0 - 12W_1W_0).$

From the last proposition, we have the following corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

Corollary 4.44. *For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k Q_k^2 = \frac{1}{16}((-1)^n (3Q_{n+3}^2 - 3Q_{n+2}^2 + 4Q_{n+1}^2 + 2Q_{n+3}Q_{n+2} - 4Q_{n+3}Q_{n+1} - 4Q_{n+2}Q_{n+1}) -$
$2).$

(b) $\sum_{k=0}^{n}(-1)^k Q_{k+1}Q_k = \frac{1}{16}((-1)^n (Q_{n+3}^2 - Q_{n+2}^2 - 4Q_{n+1}^2 - 2Q_{n+3}Q_{n+2} + 4Q_{n+3}Q_{n+1} + 4Q_{n+2}Q_{n+1}) +$
$2).$

(c) $\sum_{k=0}^{n}(-1)^k Q_{k+2}Q_k = \frac{1}{16}((-1)^n (Q_{n+3}^2 - Q_{n+2}^2 - 4Q_{n+1}^2 + 6Q_{n+3}Q_{n+2} + 4Q_{n+3}Q_{n+1} - 12$
$Q_{n+2}Q_{n+1}) - 6).$

Taking $W_n = D_n$ with $D_0 = 3, D_1 = 0, D_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Perrin numbers.

Corollary 4.45. *For $n \geq 0$, Jacobsthal-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k L_k^2 = \frac{1}{16}((-1)^n (3L_{n+3}^2 - 3L_{n+2}^2 + 4L_{n+1}^2 + 2L_{n+3}L_{n+2} - 4L_{n+3}L_{n+1} - 4L_{n+2}L_{n+1}) +$
$24).$

(b) $\sum_{k=0}^{n}(-1)^k L_{k+1}L_k = \frac{1}{16}((-1)^n (L_{n+3}^2 - L_{n+2}^2 - 4L_{n+1}^2 - 2L_{n+3}L_{n+2} + 4L_{n+3}L_{n+1} + 4L_{n+2}L_{n+1}) -$
$8).$

(c) $\sum_{k=0}^{n}(-1)^k L_{k+2}L_k = \frac{1}{16}((-1)^n (L_{n+3}^2 - L_{n+2}^2 - 4L_{n+1}^2 + 6L_{n+3}L_{n+2} + 4L_{n+3}L_{n+1} - 12L_{n+2}L_{n+1}) -$
$8).$

Taking $r = 1, s = 0, t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.46. *If $r = 1, s = 0, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{3}((-1)^n (W_{n+3}^2 - 2W_{n+2}^2 + 2W_{n+1}^2 - 2W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1}) + W_2^2 +$
$2W_0^2 - 2W_1^2 - 2W_2W_0 + 2W_1W_0).$

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{3}((-1)^n (W_{n+3}^2 + W_{n+2}^2 - W_{n+1}^2 - 3W_{n+3}W_{n+2}$
$+ W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1}) + W_2^2 + W_1^2 - W_0^2 - 3W_2W_1 + W_2W_0 + 2W_1W_0).$

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{3}((-1)^n (-3W_{n+2}W_{n+1} + 3W_{n+3}W_{n+1}) + 3W_2W_0 - 3W_1W_0).$

From the last proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 4.47. *For $n \geq 0$, Narayana numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k N_k^2 = \frac{1}{3}((-1)^n (N_{n+3}^2 - 2N_{n+2}^2 + 2N_{n+1}^2 - 2N_{n+3}N_{n+1} + 2N_{n+2}N_{n+1}) - 1).$

(b) $\sum_{k=0}^{n}(-1)^k N_{k+1}N_k = \frac{1}{3}((-1)^n (N_{n+3}^2 + N_{n+2}^2 - N_{n+1}^2 - 3N_{n+3}N_{n+2} + N_{n+3}N_{n+1} + 2N_{n+2}N_{n+1}) - 1).$

(c) $\sum_{k=0}^{n}(-1)^k N_{k+2}N_k = \frac{1}{3}(-1)^n (-3N_{n+2}N_{n+1} + 3N_{n+3}N_{n+1}).$

Taking $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

Corollary 4.48. *For $n \geq 0$, Narayana-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k U_k^2 = \frac{1}{3}((-1)^n (U_{n+3}^2 - 2U_{n+2}^2 + 2U_{n+1}^2 - 2U_{n+3}U_{n+1} + 2U_{n+2}U_{n+1}) + 17).$

(b) $\sum_{k=0}^{n}(-1)^k U_{k+1}U_k = \frac{1}{3}((-1)^n (U_{n+3}^2 + U_{n+2}^2 - U_{n+1}^2 - 3U_{n+3}U_{n+2} + U_{n+3}U_{n+1} + 2U_{n+2}U_{n+1}) - 1).$

(c) $\sum_{k=0}^{n}(-1)^k U_{k+2}U_k = \frac{1}{3}(-1)^n (-3U_{n+2}U_{n+1} + 3U_{n+3}U_{n+1}).$

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $W_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

Corollary 4.49. *For $n \geq 0$, Narayana-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k H_k^2 = \frac{1}{3}((-1)^n (H_{n+3}^2 - 2H_{n+2}^2 + 2H_{n+1}^2 - 2H_{n+3}H_{n+1} + 2H_{n+2}H_{n+1}) + 10).$

(b) $\sum_{k=0}^{n}(-1)^k H_{k+1}H_k = \frac{1}{3}((-1)^n (H_{n+3}^2 + H_{n+2}^2 - H_{n+1}^2 - 3H_{n+3}H_{n+2} + H_{n+3}H_{n+1} + 2H_{n+2}H_{n+1}) + 1).$

(c) $\sum_{k=0}^{n}(-1)^k H_{k+2}H_k = \frac{1}{3}((-1)^n (-3H_{n+2}H_{n+1} + 3H_{n+3}H_{n+1}) + 18).$

Taking $r = 1, s = 1, t = 2$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.50. *If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{15}((-1)^n (2W_{n+3}^2 - 3W_{n+2}^2 + 7W_{n+1}^2 - W_{n+3}W_{n+2} - 6W_{n+3}W_{n+1} + 4W_{n+2}W_{n+1}) + 2W_2^2 - 3W_1^2 + 7W_0^2 - W_2W_1 - 6W_2W_0 + 4W_1W_0)$

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{10}((-1)^n (W_{n+3}^2 + W_{n+2}^2 - 4W_{n+1}^2 - 3W_{n+3}W_{n+2} + 2W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1}) + W_2^2 + W_1^2 - 4W_0^2 - 3W_2W_1 + 2W_2W_0 + 2W_1W_0)$

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{30}((-1)^n (W_{n+3}^2 - 9W_{n+2}^2 - 4W_{n+1}^2 + 7W_{n+3}W_{n+2} + 12W_{n+3}W_{n+1} - 28W_{n+2}W_{n+1}) + W_2^2 - 9W_1^2 - 4W_0^2 + 7W_2W_1 + 12W_2W_0 - 28W_1W_0)$

From the above proposition, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J = 1$).

Corollary 4.51. *For $n \geq 0$, third order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k J_k^2 = \frac{1}{15}((-1)^n (2J_{n+3}^2 - 3J_{n+2}^2 + 7J_{n+1}^2 - J_{n+3}J_{n+2} - 6J_{n+3}J_{n+1} + 4J_{n+2}J_{n+1}) - 2).$

(b) $\sum_{k=0}^{n}(-1)^k J_{k+1}J_k = \frac{1}{10}((-1)^n (J_{n+3}^2 + J_{n+2}^2 - 4J_{n+1}^2 - 3J_{n+3}J_{n+2} + 2J_{n+3}J_{n+1} + 2J_{n+2}J_{n+1}) - 1).$

(c) $\sum_{k=0}^{n}(-1)^k J_{k+2}J_k = \frac{1}{30}((-1)^n (J_{n+3}^2 - 9J_{n+2}^2 - 4J_{n+1}^2 + 7J_{n+3}J_{n+2} + 12J_{n+3}J_{n+1} - 28J_{n+2}J_{n+1}) - 1).$

From the above proposition, we have the following corollary which gives sum formulas of third-order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j = 5$).

Corollary 4.52. *For $n \geq 0$, third-order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k j_k^2 = \frac{1}{15}((-1)^n (2j_{n+3}^2 - 3j_{n+2}^2 + 7j_{n+1}^2 - j_{n+3}j_{n+2} - 6j_{n+3}j_{n+1} + 4j_{n+2}j_{n+1}) + 18).$

(b) $\sum_{k=0}^{n}(-1)^k j_{k+1}j_k = \frac{1}{10}((-1)^n (j_{n+3}^2 + j_{n+2}^2 - 4j_{n+1}^2 - 3j_{n+3}j_{n+2} + 2j_{n+3}j_{n+1} + 2j_{n+2}j_{n+1}) + 19).$

(c) $\sum_{k=0}^{n}(-1)^k j_{k+2}j_k = \frac{1}{30}((-1)^n (j_{n+3}^2 - 9j_{n+2}^2 - 4j_{n+1}^2 + 7j_{n+3}j_{n+2} + 12j_{n+3}j_{n+1} - 28j_{n+2}j_{n+1}) + 99).$

Taking $r = 2, s = 3, t = 5$ in Theorem 3.1, we obtain the following Proposition.

Proposition 4.53. *If $r = 2, s = 3, t = 5$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{825}((-1)^n (19W_{n+3}^2 - 11W_{n+2}^2 + 350W_{n+1}^2 - 42W_{n+3}W_{n+2} - 170W_{n+3}W_{n+1} + 280W_{n+2}W_{n+1}) + 19W_2^2 - 11W_1^2 + 350W_0^2 - 170W_2W_0 - 42W_2W_1 + 280W_1W_0).$

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{825}((-1)^n (17W_{n+3}^2 + 77W_{n+2}^2 - 425W_{n+1}^2 - 81W_{n+3}W_{n+2} + 65W_{n+3}W_{n+1} - 10W_{n+2}W_{n+1}) + 17W_2^2 + 77W_1^2 - 425W_0^2 - 81W_2W_1 + 65W_2W_0 - 10W_1W_0).$

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{825}((-1)^n (6W_{n+3}^2 - 264W_{n+2}^2 - 150W_{n+1}^2 + 117W_{n+3}W_{n+2} + 120W_{n+3}W_{n+1} - 780W_{n+2}W_{n+1}) + 6W_2^2 - 264W_1^2 - 150W_0^2 + 117W_2W_1 + 120W_2W_0 - 780W_1W_0).$

From the last proposition, we have the following corollary which gives sum formulas of 3-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 4.54. *For $n \geq 0$, 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k G_k^2 = \frac{1}{825}((-1)^n (19G_{n+3}^2 - 11G_{n+2}^2 + 350G_{n+1}^2 - 42G_{n+3}G_{n+2} - 170G_{n+3}G_{n+1} + 280 G_{n+2}G_{n+1}) - 19).$

(b) $\sum_{k=0}^{n}(-1)^k G_{k+1}G_k = \frac{1}{825}((-1)^n (17G_{n+3}^2 + 77G_{n+2}^2 - 425G_{n+1}^2 - 81G_{n+3}G_{n+2} + 65G_{n+3}G_{n+1} - 10G_{n+2}G_{n+1}) - 17).$

(c) $\sum_{k=0}^{n}(-1)^k G_{k+2}G_k = \frac{1}{825}((-1)^n (6G_{n+3}^2 - 264G_{n+2}^2 - 150G_{n+1}^2 + 117G_{n+3}G_{n+2} + 120G_{n+3}G_{n+1} - 780G_{n+2}G_{n+1}) - 6).$

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

Corollary 4.55. *For $n \geq 0$, Lucas 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k H_k^2 = \frac{1}{825}((-1)^n (19H_{n+3}^2 - 11H_{n+2}^2 + 350H_{n+1}^2 - 42H_{n+3}H_{n+2} - 170H_{n+3}H_{n+1} + 280 H_{n+2}H_{n+1}) + 746).$

(b) $\sum_{k=0}^{n}(-1)^k H_{k+1}H_k = \frac{1}{825}((-1)^n (17H_{n+3}^2 + 77H_{n+2}^2 - 425H_{n+1}^2 - 81H_{n+3}H_{n+2} + 65H_{n+3}H_{n+1} - 10H_{n+2}H_{n+1}) - 1547).$

(c) $\sum_{k=0}^{n}(-1)^k H_{k+2}H_k = \frac{1}{825}((-1)^n (6H_{n+3}^2 - 264H_{n+2}^2 - 150H_{n+1}^2 + 117H_{n+3}H_{n+2} + 120H_{n+3}H_{n+1} - 780H_{n+2}H_{n+1}) - 546).$

From the last proposition, we have the following corollary which gives sum formulas of modified 3-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 4.56. *For $n \geq 0$, modified 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k E_k^2 = \frac{1}{825}((-1)^n (19E_{n+3}^2 - 11E_{n+2}^2 + 350E_{n+1}^2 - 42E_{n+3}E_{n+2} - 170E_{n+3}E_{n+1} + 280 E_{n+2}E_{n+1}) - 34).$

(b) $\sum_{k=0}^{n}(-1)^k E_{k+1}E_k = \frac{1}{825}((-1)^n (17E_{n+3}^2 + 77E_{n+2}^2 - 425E_{n+1}^2 - 81E_{n+3}E_{n+2} + 65E_{n+3}E_{n+1} - 10E_{n+2}E_{n+1}) + 13).$

(c) $\sum_{k=0}^{n}(-1)^k E_{k+2}E_k = \frac{1}{825}((-1)^n (6E_{n+3}^2 - 264E_{n+2}^2 - 150E_{n+1}^2 + 117E_{n+3}E_{n+2} + 120E_{n+3}E_{n+1} - 780E_{n+2}E_{n+1}) - 141).$

4.3 The case $x = 1 + i$

In this subsection we consider the special case $x = 1 + i$.

Taking $x = 1 + i, r = s = t = 1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 4.57. *If $x = 1 + i, r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(1+i)^k W_k^2 = \frac{1}{-7+28i}((1+i)^n\,((6+14i)W_{n+3}^2 + (28+22i)W_{n+2}^2 + (27-i)W_{n+1}^2 + (-24-32i)W_{n+3}W_{n+2} + (-8-16i)W_{n+3}W_{n+1} + (-8+16i)W_{n+2}W_{n+1}) - (10+4i)\,W_2^2 - (25-3i)\,W_1^2 - (13-14i)\,W_0^2 + (28+4i)\,W_2 W_1 + (12+4i)\,W_2 W_0 - (4+12i)\,W_1 W_0).$

(b) $\sum_{k=0}^{n}(1+i)^k W_{k+1} W_k = \frac{1}{-7+28i}((1+i)^n\,((-6-2i)W_{n+3}^2 + (-16-8i)W_{n+2}^2 + (-4-8i)W_{n+1}^2 + (18+12i)W_{n+3}W_{n+2} + (-2+14i)W_{n+3}W_{n+1} + (11-25i)W_{n+2}W_{n+1}) + (4-2i)\,W_2^2 + (12-4i)\,W_1^2 + (6+2i)\,W_0^2 - (15-3i)\,W_2 W_1 - (6+8i)\,W_2 W_0 + (7+18i)\,W_1 W_0).$

(c) $\sum_{k=0}^{n}(1+i)^k W_{k+2} W_k = \frac{1}{-7+28i}((1+i)^n\,((-4+4i)W_{n+3}^2 + (4-10i)W_{n+2}^2 - 8W_{n+1}^2 + 10i W_{n+3}W_{n+2} + (9-11i)W_{n+3}W_{n+1} + (4-2i)W_{n+1}W_{n+2}) - 4i W_2^2 + (4-4i)\,W_0^2 + (3+7i)\,W_1^2 - (5+5i)\,W_2 W_1 + (1+10i)\,W_2 W_0 + (-1+3i)\,W_1 W_0).$

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 4.58. *For $n \geq 0$, Tribonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n}(1+i)^k T_k^2 = \frac{1}{-7+28i}((1+i)^n\,((6+14i)T_{n+3}^2 + (28+22i)T_{n+2}^2 + (27-i)T_{n+1}^2 + (-24-32i)T_{n+3}T_{n+2} + (-8-16i)T_{n+3}T_{n+1} + (-8+16i)T_{n+2}T_{n+1}) - 7 + 3i).$

(b) $\sum_{k=0}^{n}(1+i)^k T_{k+1} T_k = \frac{1}{-7+28i}((1+i)^n\,((-6-2i)T_{n+3}^2 + (-16-8i)T_{n+2}^2 + (-4-8i)T_{n+1}^2 + (18+12i)T_{n+3}T_{n+2} + (-2+14i)T_{n+3}T_{n+1} + (11-25i)T_{n+2}T_{n+1}) + 1 - 3i).$

(c) $\sum_{k=0}^{n}(1+i)^k T_{k+2} T_k = \frac{1}{-7+28i}((1+i)^n\,((-4+4i)T_{n+3}^2 + (4-10i)T_{n+2}^2 - 8T_{n+1}^2 + 10i T_{n+3}T_{n+2} + (9-11i)T_{n+3}T_{n+1} + (4-2i)T_{n+1}T_{n+2}) - 2 - 2i).$

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 4.59. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(1+i)^k K_k^2 = \frac{1}{-7+28i}((1+i)^n\,((6+14i)K_{n+3}^2 + (28+22i)K_{n+2}^2 + (27-i)K_{n+1}^2 + (-24-32i)K_{n+3}K_{n+2} + (-8-16i)K_{n+3}K_{n+1} + (-8+16i)K_{n+2}K_{n+1}) - 52 + 105i).$

(b) $\sum_{k=0}^{n}(1+i)^k K_{k+1} K_k = \frac{1}{-7+28i}((1+i)^n\,((-6-2i)K_{n+3}^2 + (-16-8i)K_{n+2}^2 + (-4-8i)K_{n+1}^2 + (18+12i)K_{n+3}K_{n+2} + (-2+14i)K_{n+3}K_{n+1} + (11-25i)K_{n+2}K_{n+1}) + 24 - 13i).$

(c) $\sum_{k=0}^{n}(1+i)^k K_{k+2} K_k = \frac{1}{-7+28i}((1+i)^n\,((-4+4i)K_{n+3}^2 + (4-10i)K_{n+2}^2 - 8K_{n+1}^2 + 10i K_{n+3}K_{n+2} + (9-11i)K_{n+3}K_{n+1} + (4-2i)K_{n+1}K_{n+2}) + 30 + 19i).$

Corresponding sums of the other third order generalized Tribonacci numbers can be calculated similarly.

5 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Tribonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Tribonacci, Tribonacci-Lucas, Padovan, Perrin

numbers and the other third order recurrence relations. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. Our future work will be investigation of the closed forms of the sum formulas for the squares of generalized Tetranacci numbers.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

[1] Bruce I. A modified Tribonacci Sequence. Fibonacci Quarterly. 1984;22(3):244-246.

[2] Catalani M. Identities for Tribonacci-related sequences; 2012. arXiv:math/0209179

[3] Choi E. Modular Tribonacci numbers by matrix method. Journal of the Korean Society of Mathematical Education Series B: Pure and Applied. Mathematics. 2013;20(3):207-221.

[4] Elia M. Derived sequences. The Tribonacci recurrence and cubic forms. Fibonacci Quarterly. 2001;39(2):107-115.

[5] Lin PY. De moivre-type identities for the Tribonacci numbers. Fibonacci Quarterly. 1988;26:131-134.

[6] Pethe S. Some identities for Tribonacci sequences. Fibonacci Quarterly. 1988;26(2):144-151.

[7] Scott A, Delaney T, Hoggatt Jr. V. The Tribonacci sequence. Fibonacci Quarterly. 1977;15(3):193-200.

[8] Shannon AG, Horadam AF. Some properties of third-order recurrence relations. The Fibonacci Quarterly. 1972;10(2):135-146.

[9] Shannon A. Tribonacci numbers and Pascal's pyramid. Fibonacci Quarterly. 1977;15(3):268-275.

[10] Soykan Y. Tribonacci and Tribonacci-Lucas sedenions. Mathematics. 2019;7(1):74.

[11] Spickerman W. Binet's formula for the Tribonacci sequence. Fibonacci Quarterly. 1982;20:118-120.

[12] Yalavigi CC. A note on 'another generalized fibonacci sequence. The Mathematics Student. 1971;39:407-408.

[13] Yalavigi CC. Properties of Tribonacci numbers. Fibonacci Quarterly. 1972;10(3):231-246.

[14] Yilmaz N, Taskara N. Tribonacci and Tribonacci-Lucas numbers via the determinants of special Matrices. Applied Mathematical Sciences. 2014;8(39):1947-1955.

[15] Marcellus E. Waddill. Using matrix techniques to establish properties of a generalized tribonacci sequence (in Applications of Fibonacci Numbers, Bergum GE, et al. Eds.). Kluwer Academic Publishers. Dordrecht, The Netherlands. 1991;4:299-308.

[16] Sloane NJA. The on-line encyclopedia of integer sequences. Available:http://oeis.org/

[17] Soykan Y. On generalized grahaml numbers. Journal of Advances in Mathematics and Computer Science. 2020;35(2):42-57.
DOI: 10.9734/JAMCS/2020/v35i230248

[18] Čerin Z. Formulae for sums of Jacobsthal–Lucas numbers. Int. Math. Forum. 2007;2(40):1969-1984.

[19] Čerin, Z. Sums of squares and products of jacobsthal numbers. Journal of Integer Sequences. 2007;10.

[20] Gnanam A, Anitha B. Sums of squares jacobsthal numbers. IOSR Journal of Mathematics. 2015;11(6):62-64.

[21] Kiliç E, Taşçi D. The linear algebra of the pell matrix. Boletín de la Sociedad Matemática Mexicana. 2005;3(11).

[22] Kılıc E. Sums of the squares of terms of sequence $\{u_n\}$, Proc. Indian Acad. Sci. (Math. Sci.) 2008;118(1):27-41.

[23] Soykan Y. Closed formulas for the sums of squares of generalized fibonacci numbers. Asian Journal of Advanced Research and Reports. 2020;9(1):23-39.
Available:https://doi.org/10.9734/ajarr/2020-/v9i130212

[24] Frontczak R. Sums of cubes over odd-index fibonacci numbers. Integers. 2018;18.

[25] Soykan Y. Closed formulas for the sums of cubes of generalized fibonacci numbers: Closed formulas of and $\sum_{k=0}^{n} W_k^3$ and $\sum_{k=1}^{n} W_{-k}^3$. Archives of Current Research International. 2020;20(2):58-69.
DOI:10.9734/ACRI/2020/v20i230177

[26] Soykan Y. A study on sums of cubes of generalized fibonacci numbers: Closed formulas of $\sum_{k=0}^{n} x^k W_k^3$ and $\sum_{k=1}^{n} x^k W_{-k}^3$. Preprints. 2020;2020040437.
DOI: 10.20944/preprints202004.0437.v1

[27] Soykan Y. On sums of cubes of generalized fibonacci numbers: Closed formulas of $\sum_{k=0}^{n} k W_k^3$ and $\sum_{k=1}^{n} k W_{-k}^3$. Asian Research Journal of Mathematics. 2020;16(6):37-52.
DOI:10.9734/ARJOM/2020/v16i630196

[28] Wamiliana, Suharsono, Kristanto PE. Counting the sum of cubes for lucas and gibonacci numbers. Science and Technology Indonesia. 2019;4(2):31-35.

[29] Chen L, Wang X. The power sums involving fibonacci polynomials and their applications. Symmetry. 2019;11.
DOI.org/10.3390/sym11050635

[30] Frontczak R. Sums of powers of Fibonacci and Lucas numbers: A new bottom-up Approach. Notes on Number Theory and Discrete Mathematics. 2018;24(2):94-103.

[31] Prodinger H. Sums of powers of fibonacci polynomials. Proc. Indian Acad. Sci. (Math. Sci.). 2009;119(5):567-570.

[32] Raza Z, Riaz M, Ali MA. Some inequalities on the norms of special matrices with generalized tribonacci and generalized pell-padovan sequences; 2015.
Available:http://arxiv.org/abs/1407.1369v2

[33] Soykan Y. A closed formula for the sums of squares of generalized tribonacci numbers. Journal of Progressive Research in Mathematics. 2020;16(2):2932-2941.

[34] Prodinger H, Selkirk SJ. Sums of squares of tetranacci numbers: A generating function approach; 2019.
Available:http://arxiv.org/abs/1906.08336v1

[35] Schumacher R. How to sum the squares of the Tetranacci numbers and the Fibonacci m-step numbers. Fibonacci Quarterly. 2019;57:168-175.

[36] Soykan Y. Simson identity of generalized m-step fibonacci numbers. Int. J. Adv. Appl. Math. and Mech. 2019;7(2):45-56.

[37] Davis PJ. Circulant matrices. John Wiley & Sons, New York; 1979.

[38] Shen S, Cen J. On the bounds for the norms of r-circulant matrices with the fibonacci and lucas numbers. Applied Mathematics and Computation. 2010;216:2891-2897.

[39] Kızılateş C, Tuglu N. On the bounds for the spectral norms of geometric circulant matrices. Journal of Inequalities and Applications. 2016(1). Article ID 312, 2016.
DOI 10.1186/s13660-016-1255-1

[40] Polatlı E. On the bounds for the spectral norms of r-circulant matrices with a type of catalan triangle numbers. Journal of Science and Arts. 2019;3(48):575-578.

[41] He C, Ma J, Zhang K, Wang Z. The upper bound estimation on the spectral norm of r-circulant matrices with the fibonacci and lucas numbers. J. Inequal. Appl. 2015:72:2015.

[42] Merikoski JK, Haukkanen P, Mattila M, Tossavainen T. On the spectral and frobenius norm of a generalized fibonacci r-circulant matrix. Special Matrices. 2018;6:23-36.

[43] Raza Z, Ali MA. On the norms of some special matrices with generalized fibonacci sequence. J. Appl. Math. & Informatics. 2015;33(5-6):593-605.

[44] Shen S. On the norms of circulant matrices with the (k,h)-fibonacci and (k,h)-lucas numbers. Int. J. Contemp. Math. Sciences. 2011;6(18):18:887-894.

[45] Shen S. The spectral norms of circulant matrices involving (k,h)-fibonacci and (k,h)-lucas numbers. Int. J. Contemp. Math. Sciences. 2014;9(14):661-665.

[46] Shen S, Cen J. On the spectral norms of r-circulant matrices with the k-fibonacci and k-lucas numbers. Int. J. Contemp. Math. Sciences. 2010;5(12):569-578.

[47] Shi B. The spectral norms of geometric circulant matrices with the generalized k-horadam numbers. Journal of Inequalities and Applications. 2018:14:2018. available:https://doi.org/10.1186/s13660-017-1608-4

[48] Solak S. On the norms of circulant matrices with the fibonacci and lucas numbers. Applied Mathematics and Computation. 2005;160:125-132.

[49] Solak S. Erratum to "on the norms of circulant matrices with the fibonacci and lucas numbers" [Appl. Math. Comput. 2005;160:125-132. Applied Mathematics and Computation. 2007;190:1855-856.

[50] Tuglu N, Kızılateş C. On the norms of circulant and r-circulant matrices with the hyperharmonic fibonacci numbers. Journal of Inequalities and Applications; 2015. Article ID 253
Available:http://dx.doi.org/10.1186/s13660-015-0778-1

[51] Alptekin E. Pell G. Pell-Lucas ve Modified Pell sayıları ile tanımlı circulant ve semicirculant matrisler (Doctoral dissertation, Selçuk Üniversitesi Fen Bilimleri Enstitüsü); 2005.

[52] Turkmen R, GökbaşH. On the spectral norm of r-circulant matrices with the pell and pell-lucas numbers. J. Inequal. Appl. 2016;65:2016.

[53] Radicic B. On k-circulant matrices involving the jacobsthal numbers. Revista De La Union Matematica Argentina. 2019;60(2):431-442.

[54] Uygun Ş. Some bounds for the norms of circulant matrices with the k-jacobsthal and k-jacobsthal lucas numbers. Journal of Mathematics Research. 2016;8(6):133-138.

[55] Uygun Ş, YaşamalıS. On the bounds for the norms of circulant matrices with the jacobsthal and jacobsthal-lucas numbers. Notes on Number Theory and Discrete Mathematics. 2017;23(1):91-98.

[56] Uygun Ş, YaşamalıS. On the bounds for the norms of r-circulant matrices with the jacobsthal and jacobsthal–lucas numbers. International Journal of Pure and Applied Mathematics. 2017;112(1):93-102.

[57] Kızılateş C, Tuglu N. On the norms of geometric and symmetric geometric circulant matrices with the tribonacci number. Gazi University Journal of Science. 2018;31(2):555-567.

[58] Raza Z, Ali MA. On the Norms of Circulant, r-Circulant, Semi-Circulant and Hankel Matrices with Tribonacci Sequence; 2014.
Available:http://arxiv.org/abs/1407.1369v1

[59] Coskun A, Taskara N. On the some properties of circulant matrices with third order linear recurrent sequences. Mathematical Sciences and Applications E-Notes. 2018;6(1):12-18.

[60] Pacheenburawana A, Sintunavarat W. On the spectral norms of r-circulant matrices with the padovan and perrin sequences. Journal of Mathematical Analysis. 2018;9(3):110-122.

[61] Sintunavarat, W. The upper bound estimation for the spectral norm of r-circulant and symmetric r-circulant matrices with the padovan sequence. J. Nonlinear Sci. Appl. 2016;9:92-101.

[62] Özkoç A, Ardıyok E. Circulant and negacyclic matrices via tetranacci numbers. Honam Mathematical J. 2016;38(4):725-738.

[63] Gökbaş H, Köse H. Some sum formulas for products of pell and pell-lucas numbers. Int. J. Adv. Appl. Math. and Mech. 2017;4(4):1-4.

[64] Koshy T. Fibonacci and lucas numbers with applications. A Wiley-Interscience Publication, New York; 2001.

[65] Koshy T. Pell and pell-lucas numbers with applications. Springer, New York; 2014.

[66] Hansen RT. General identities for linear fibonacci and lucas summations. Fibonacci Quarterly. 1978;16(2):121-28.

[67] Soykan Y. On summing formulas for generalized fibonacci and gaussian generalized fibonacci numbers. Advances in Research. 2019;20(2):1-15.

[68] Soykan Y. Corrigendum: On summing formulas for generalized fibonacci and gaussian generalized fibonacci numbers; 2020.
Available:https://www.researchgate.net/publi-cation/337063487_Corrigendum_On_Summing_Formulas _For_Generalized_Fibonacci_and_Gaussian_Generalized_Fibonacci_Numbers

[69] SoykanY. On summing formulas for horadam numbers. Asian Journal of Advanced Research and Reports. 2020;8(1):45-61.
DOI: 10.9734/AJARR/2020/v8i130192

[70] Soykan Y. Generalized fibonacci numbers: Sum formulas. Journal of Advances in Mathematics and Computer Science. 2020;35(1):89-104.
DOI:10.9734/JAMCS/2020/v35i130241.

[71] Frontczak R. Sums of tribonacci and tribonacci-lucas numbers. International Journal of Mathematical Analysis. 2018;12(1):19-24.

[72] Parpar T. k'ncı Mertebeden Rekürans Bağıntısının Özellikleri ve Bazı Uygulamaları, Selçuk Üniversitesi, Fen Bilimleri Enstitüsü, Yüksek Lisans Tezi; 2011.

[73] Soykan Y. Summing formulas for generalized tribonacci numbers. Universal Journal of Mathematics and Applications. 2020;3(1):1-11.
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.637876

[74] Soykan Y. Matrix sequences of tribonacci and tribonacci-lucas numbers; 2018.
arXiv:1809.07809v1 [math.NT]

[75] Soykan. Summation formulas for generalized tetranacci numbers. Asian Journal of Advanced Research and Reports. 2019;7(2):1-12.
DOI.org/10.9734/ajarr/2019/v7i230170

[76] Waddill ME. The Tetranacci sequence and generalizations. Fibonacci Quarterly. 1992;9-20.

[77] Soykan Y. Linear summing formulas of generalized pentanacci and gaussian generalized pentanacci numbers. Journal of Advanced in Mathematics and Computer Science. 2019;33(3):1-14.

[78] Soykan Y. Sum formulas for generalized fifth-order linear recurrence sequences. Journal of Advances in Mathematics and Computer Science. 2019;34(5):1-14.
Article no.JAMCS.53303
ISSN: 2456-9968
DOI: 10.9734/JAMCS/2019/v34i530224

[79] Soykan Y. On summing formulas of generalized hexanacci and gaussian generalized hexanacci numbers. Asian Research Journal of Mathematics. 2019;14(4):1-14.
Article no.ARJOM.50727