COMPARATIVE ANALYSIS OF THE INVERSE WEIBULL DISTRIBUTION BY USING TYPE-II CENSORED SAMPLE

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AUTHORS' CONTRIBUTIONS

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

Parameter estimation become complicated when censoring is present in the sample. Some time it is not possible to give a mathematical expression about estimated values of parameters in Maximum Likelihood (ML) method. In this situation iteration method is used, to find estimated values of parameters in numeric form . There are several Modified Maximum Likelihood (MML) estimation procedures which provide a mathematical expression about parametric value. For the two-parameter inverse Weibull distribution by using type II censored sample, assuming fixed shape parameter, a modified maximum likelihood estimator is proposed by [1] in which they use a simple approximation for intractable terms to estimate the scale parameter. In this paper a comparative study of the proposed estimator [1] is made with MML estimator of [2] ML estimators of [3].

Keywords: Inverse weibull distribution; Type II censored sample; Type II censoring; Modified maximum likelihood estimator; order statistics; Bias; Asymptotic variance; Mean square error.

1. INTRODUCTION

The probability density function of inverse Weibull distribution is given as:

$$f(y) = \frac{m}{\theta} \frac{1}{y^{m+1}} \exp\left(-\frac{1}{\theta\, y^m}\right) \qquad , y > 0, \; m, \; \theta > 0. \tag{1.1}$$

$= 0$ other wise .

and the corresponding distribution function is:

$$F(y) = \exp\left(-\frac{1}{\theta\, y^m}\right) \tag{1.2}$$

Where "m" is shape and "θ" is scale parameter

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2. THE MODIFIED MAXIMUM LIKELIHOODESTIMATOR (MMLE) OF THE SCALE PARAMETER OF THE INVERSE WEIBULL DISTRIBUTION

For the doubly type ll censored sample with r samples censored on the left and s samples
censored on the right .The likelihood function is given as

$$L = \frac{n!}{r!s!}\left[F(Y_{r+1})\right]^r \left[1 - F(Y_{n-s})\right]^s \prod_{i=r+1}^{n-s} f(y_i) \tag{2.1}$$

Where $r = [nq_1] + 1$ and $s = [nq_2] + 1$, q_1 is the proportion of left censored sample and q_2 is the proportion of right censored sample .By using (1.1)and(1.2) in (2.1) we get

$$L = \frac{n!}{r!s!}\left[\left(\exp\left(-\frac{1}{\theta\, y_{r+1}{}^m}\right)\right)\right]^r \left[1 - \exp\left(-\frac{1}{\theta\, y_{n-s}{}^m}\right)\right]^s \prod_{i=r+1}^{n-s}\left[\frac{m}{\theta}\frac{1}{y_i{}^{m+1}}\exp\left(-\frac{1}{\theta\, y_i{}^m}\right)\right]$$

The first derivatives of the log-likelihood function with respect to θ is given by

$$\frac{\partial \ln L}{\partial \theta} = r\left(\frac{1}{\theta^2\, y_{r+1}{}^m}\right) + \frac{s}{\theta^2\, y_{n-s}{}^m}\left[-\frac{\exp\left(-\frac{1}{\theta\, y_{n-s}{}^m}\right)}{1 - \exp\left(-\frac{1}{\theta\, y_{n-s}{}^m}\right)}\right] - \frac{(n-s-r)}{\theta} + \frac{1}{\theta^2}\sum_{i=r+1}^{n-s}\frac{1}{y^m{}_i} \tag{2.2}$$

Putting $z_{n-s} = \theta y^m{}_{n-s}$ in equation (2.2) we get

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{\theta^2 y^m{}_{r+1}} + \frac{s}{\theta^2 y^m{}_{n-s}}\left(\frac{-\exp\left(-\frac{1}{z_{n-s}}\right)}{1 - \exp\left(-\frac{1}{z_{n-s}}\right)}\right) - \frac{(n-s-r)}{\theta} + \frac{1}{\theta^2}\sum_{i=r+1}^{n-s}\frac{1}{y^m{}_i} \overset{(set)}{=} 0 \tag{2.3}$$

By using $\exp(z^{-1}) = \dfrac{1 + \dfrac{1}{2z}}{1 - \dfrac{1}{2z}}$ for intractable term in (2.3), see [4]. Solving (2.3) for θ ,the modified maximum likelihood estimator of θ is given as

$$\theta^\wedge = \frac{\dfrac{r}{y_{r+1}{}^m} + \dfrac{s}{2y^m{}_{n-s}} + \sum_{i=r+1}^{n-s}\dfrac{1}{y^m{}_i}}{n - r} \tag{2.4}$$

3. ASYMPTOTIC VARIANCE AND BIAS OF $\theta^\wedge$

By using Glivenko-Cantelli lemma given by [5],
We have

$$p_1 = G^{-1}(q_1) \text{ and } p_2 = G^{-1}(1 - q_2) \text{ So as } n \to \infty$$

$$z_{r+1} = G^{-1}(q_1) \quad \text{and} \quad z_{n-s} = G^{-1}(1-q_2)$$

z_{r+1} can be evaluated by using following relation

$$G(z_{r+1}) = q_1$$

$$\int_0^{z_{r+1}} f(z)dz = q_1$$

$$z_{r+1} = -\frac{1}{\ln q1} \tag{3.1}$$

and

$$G(z_{n-s}) = 1 - q_2$$

$$\int_{z_{n-s}}^{\infty} f(z)dz = q_2$$

$$z_{n-s} = -\frac{1}{\ln(1-q_2)} \tag{3.2}$$

and

$$\lim_{n \to \infty} E\left\{ \frac{1}{n} \sum_{i=r+1}^{n-s} \frac{1}{z_i} \right\} = \int_{z_{r+1}}^{z_{n-s}} \frac{1}{z} f(z)dz = \int_{z_{r+1}}^{z_{n-s}} \frac{1}{z^3} \exp\left(-\frac{1}{z}\right)dz$$

$$= 1 - \ln(1-q_2) - q_2 + q_2 \ln(1-q_2) - q_1 + q_1 \ln q_1 \tag{3.3}$$

From the equation (2.2) we have

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{\theta^2 y^m{}_{r+1}} + \frac{s}{2\theta^2 y^m{}_{n-s}} - \frac{s}{\theta} - \frac{(n-s-r)}{\theta} + \frac{\sum_{i=r+1}^{n-s} \frac{1}{y^m{}_i}}{\theta^2}$$

Partially differentiate the above equation w.r.t θ

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{2r}{\theta^3 y^m{}_{r+1}} - \frac{2s}{2\theta^3 y^m{}_{n-s}} + \frac{s}{\theta^2} + \frac{(n-s-r)}{\theta^2} - \frac{2\sum_{i=r+1}^{n-s} \frac{1}{y^m{}_i}}{\theta^3}$$

Putting $z_{r+1} = y^m{}_{r+1}\theta, z_{n-s} = y^m{}_{n-s}\theta, z_i = y^m{}_i\theta$, we have

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{2r}{z_{r+1}\theta^2} - \frac{s}{z_{n-s}\theta^2} + \frac{s}{\theta^2} + \frac{(n-s-r)}{\theta^2} - \frac{2\sum_{i=r+1}^{n-s} \frac{1}{z_i}}{\theta^2} \tag{3.4}$$

By using equation (3.1) and (3.2) we obtain as:

$$-E\left(\frac{\partial^2 \ln L}{\partial\theta^2}\right) = \frac{n}{\theta^2}\left(-2q_1 \ln q_1 - q_2 \ln(1-q_2) - 1 + q_1 + 2E\left(\frac{1}{n}\sum_{i=r+1}^{n-s}\frac{1}{z_i}\right)\right)$$

From equation (3.3)

$$= \frac{n}{\theta^2}\left[-2q_1 \ln q_1 - q_2 \ln(1-q_2) - 1 + q_1 + 2 - 2\ln(1-q_2) - 2q_2 + 2q_2 \ln(1-q_2) - 2q_1 + 2q_1 \ln q_1\right]$$

So , Asymptotic variance is given as

$$\text{var}(\theta^\wedge) = \frac{\theta^2}{n\left[1 - q_1 - 2q_2 - 2\ln(1-q_2) + q_2 \ln(1-q_2)\right]} \tag{3.5}$$

Where $\qquad \therefore q_1 = \frac{r}{n} \therefore q_2 = \frac{s}{n}$

From (3.5) we observe as $n \to \infty$, $\text{var}(\theta^\wedge) \to 0$

Now put $\quad z_{r+1} = y^m{}_{r+1}\theta$, $z_{n-s} = y^m{}_{n-s}\theta$ and $z_i = y^m{}_i\theta$ in the equation (2.4) ,we get

$$\theta^\wedge = \frac{\dfrac{\theta r}{z_{r+1}} + \dfrac{s\theta}{2z_{n-s}} + \theta\sum_{i=r+1}^{n-s}\dfrac{1}{z_i}}{n-r}$$

Apply expectation on both sides and from the equation (3.1), (3.2) and (3.3) we have

$$Bias = E(\theta^\wedge) - \theta = \frac{\theta\left(\frac{1}{2}q_2 \ln(1-q_2) - \ln(1-q_1) - q_2\right)}{1-q_1} \tag{3.6}$$

From (3.6) we observe as sample size increases, The Bias decreases i.e. $Bias \to 0$ for large sample size

4. ESTIMATION OF THE MEAN SQUARE ERROR

From [6] for moments of order statistics from inverse weibull distribution, we have

$$\mu_{r:n}^{(k-m)} = E\left(y_{r:n}^{k-m}\right) \quad \text{then}$$

$$\mu_{r:n}^{(k-m)} = \frac{C_{r:n}\Gamma\left(2 - \dfrac{k}{m}\right)}{\theta^{\frac{k}{m}-1}}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j (a_j)^{\frac{k}{m}-2} \tag{4.1}$$

Where $\qquad C_{r:n} = \dfrac{n!}{(r-1)!(n-r)!}$

Putting k=0 in (4.1) we obtain

$$\mu_{r:n}^{(-m)} = E\left(y_{r:n}^{-m}\right)$$

$$E\left(\frac{1}{y_{r:n}^{m}}\right) = \theta \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j}(-1)^{j}\left(\frac{1}{r+j}\right)^{2} \qquad (4.2)$$

Where $\qquad \alpha_r = \dfrac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j}(-1)^{j}\left(\dfrac{1}{r+j}\right)^{2}$

And by putting k=-m in (4.1), we have

$$\mu_{r:n}^{(-m-m)} = E\left(y_{r:n}^{-2m}\right) = E\left(\frac{1}{y_{r:n}^{m}}\right)^{2}$$

$$E\left(\frac{1}{y_{r:n}^{m}}\right)^{2} = 2\theta^{2}\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^{j}\left(\frac{1}{r+j}\right)^{3} \qquad (4.3)$$

By using (4.2 and (4.3),we have)

$$\text{var}\left(\frac{1}{y^{m}_{r}}\right) = \theta^{2}\beta_{r} \qquad (4.4)$$

Where $\beta_r = 2\dfrac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^{j}\left(\dfrac{1}{r+j}\right)^{3} - \left(\alpha_r\right)^{2}$

Applying expectation on equation (2.3) and then by using equation (4.2), we obtain

$$Bias\left(\theta^{\wedge}\right) = E\left(\theta^{\wedge}\right) - \theta = \frac{\theta\left(q_1\alpha_{r+1} + \dfrac{q_2}{2}\alpha_{n-s} + \dfrac{\alpha}{n} - 1 + q_1\right)}{1 - q_1} \qquad (4.5)$$

Where $\alpha = \sum\limits_{i=r+1}^{n-s}\alpha_i$

Now applying variance on equation (2.3), then by using equation (4.4) we obtained

$$Var\left(\theta^{\wedge}\right) = \frac{\theta^{2}\left(q^2{}_1\beta_{r+1} + \dfrac{q^2{}_2}{4}\beta_{n-s} + \dfrac{\beta}{n^2}\right)}{\left(1 - q_1\right)^{2}} \qquad (4.6)$$

Where $\beta = \sum\limits_{i=r+1}^{n-s}\beta_i$

By using equation (4.5) and (4.6),we have

$$MSE\left(\theta^{\wedge}\right) = \frac{\theta^{2}}{\left(1 - q_1\right)^{2}}\left(\left(q_1\alpha_{r+1} + \frac{q_2}{2}\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1\right)^{2} + \left(q^2{}_1\beta_{r+1} + \frac{q^2{}_2}{4}\beta_{n-s}\frac{\beta}{n^2}\right)\right) \quad (4.7)$$

5. EMPIRICAL STUDY

5.1 The Modified Maximum Likelihoodestimator (MMLE) of mehrotra and Nanda

Mehrotra and Nanda [2] used the approach of MML, by replacing the hazard function in likelihood equation with its expectation in case of right Type ll censored sample of normal and gamma distribution. The reference [7] used the Mehrotra and Nanda [2] MML technique .They replace the intractable term of likelihood equations with its expectation, to get the estimators of location parameter by considering the scale parameter equal to of logistic distribution in right Type ll censored sample.

By using Mehrotra and Nanda [2] for intractable term in (2.3) the modified maximum likelihood estimator of θ is given as

$$\theta^{\wedge} = \frac{\dfrac{r}{y^m_{r+1}} + \dfrac{(s-n)}{y^m_{n-s}} + \sum_{i=r+1}^{n-s} \dfrac{1}{y^m_i}}{n-s-r} \tag{5.1}$$

And

$$E\left(\theta^{\wedge}\right) = \theta$$

And asymptotic Variance of $\theta^{\wedge}$ is given as

$$\mathrm{var}\left(\theta^{\wedge}\right) = \frac{\theta^2}{n\left(1-q_1-q_2\right)} \tag{5.2}$$

From (5.2) we observe as $n \to \infty$, $\mathrm{var}\left(\theta^{\wedge}\right) \to 0$

5.2 Estimation of the Mean Square Error

From (5.1) by using results in (4.2) and (4.4) we have

$$Bias\left(\theta^{\wedge}\right) = E\left(\theta^{\wedge}\right) - \theta = \frac{\theta\left(q_1\alpha_{r+1} + (q_2-1)\alpha_{n-s} + \dfrac{\alpha}{n} - 1 + q_1 + q_2\right)}{1-q_1-q_2} \tag{5.2.1}$$

$$\mathrm{var}\left(\theta^{\wedge}\right) = \frac{\theta^2\left(q_1^{\ 2}\beta_{r+1} + (q_2-1)^2\beta_{n-s} + \dfrac{\beta}{n^2}\right)}{\left(1-q_1-q_2\right)^2} \tag{5.2.2}$$

From the equation (5.2.1) and (5.2.2) we obtain MSE as:

$$MSE(\theta) = \frac{\theta^2}{\left(1-q_1-q_2\right)^2}\left(\left(q_1\alpha_{r+1} + (q_2-1)\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1 + q_2\right)^2 + \left(q_1^{\ 2}\beta_{r+1} + (q_2-1)^2\beta_{n-s} + \frac{\beta}{n^2}\right)\right) \tag{5.2.3}$$

5.3 ML Estimator of Kambo

The explicit solution for ML estimator of two parameter exponential distribution in the case of doubly type ll censored sample.

In this section ML estimator are given for two parameters inverse weibull distribution in the case of doubly type ll censored sample by [3] approach.

By using [3] put (3.2) in (2.3) then the maximum likelihood estimator of θ is given as

$$\theta^{\wedge} = \frac{\left(\dfrac{r}{y^m_{r+1}}\right) - \dfrac{s}{y^m_{n-s}}\left(\dfrac{1-q_2}{q_2}\right) + \displaystyle\sum_{i=r+1}^{n-s}\dfrac{1}{y^m_i}}{n-r-s} \tag{5.4}$$

and $E(\theta^{\wedge}) = \theta$

and asymptotic Variance of $\theta^{\wedge}$ is given as

$$\operatorname{var}(\theta^{\wedge}) = \frac{\theta^2}{n[1-q_1-q_2]} \tag{5.5}$$

From (5.5) we observe as $n \to \infty$, $\operatorname{var}(\theta^{\wedge}) \to 0$

5.4 Estimation of the Mean Square Error

From (5.4) by using results in (4.2) and (4.4) we have

$$Bias(\theta^{\wedge}) = E(\theta^{\wedge}) - \theta = \frac{\left(q_1\alpha_{r+1} - (1-q_2)\alpha_{n-s} + \dfrac{\alpha}{n} - 1 + q_1 + q_2\right)}{(1-q_1-q_2)} \tag{5.5.1}$$

$$\operatorname{var}(\theta^{\wedge}) = \frac{\left(q_1{}^2\beta_{r+1} + \beta_{n-s}(1-q_2)^2 + \dfrac{\beta}{n^2}\right)}{(1-q_1-q_2)^2} \tag{5.5.2}$$

From the equation (5.5.1) and (5.5.2) we obtain MSE as:

$$MSE(\theta) = \frac{\theta^2}{(1-q_1-q_2)^2}\left(\left(q_1\alpha_{r+1} + (1-q_2)\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1 + q_2\right)^2 + \left(q_1{}^2\beta_{r+1} + (1-q_2)^2\beta_{n-s} + \frac{\beta}{n^2}\right)\right) \tag{5.5.3}$$

Corollary: By substituting the value of q_2 in the equation (5.4) then the scale parameter obtained by [3] approach become scale parameter of [2].

Proof: Substitute the value of $q_2 = \dfrac{s}{n}$ in the equation (5.4), we get

$$\theta^{\wedge} = \frac{\left(\dfrac{r}{y^m_{r+1}}\right) - \dfrac{s}{y^m_{n-s}}\left(\dfrac{1-\dfrac{s}{n}}{\dfrac{s}{n}}\right) + \displaystyle\sum_{i=r+1}^{n-s}\dfrac{1}{y^m_i}}{n-r-s}$$

$$\theta^\wedge = \frac{\left(\dfrac{r}{y^m_{r+1}}\right) + \dfrac{(s-n)}{y^m_{n-s}} + \sum\limits_{i=r+1}^{n-s} \dfrac{1}{y^m_i}}{n-r-s}$$

Equation (5.4) and (5.1) are same , this establishes Corollary.

Table 1. The asymptotic variance of MML estimates from doubly censored sample in term of θ^2 for n=10,20,30,50 and 100

q1	q2	n=10 $\mathrm{var}\!\left(\theta^\wedge\right)p/\theta^2$	n=10 $\mathrm{var}\!\left(\theta^\wedge\right)k/\theta^2$	n=10 $\mathrm{var}\!\left(\theta^\wedge\right)m\,\&\,n/\theta^2$
0	0	0.1	0.1	0.1
0	0.1	0.1	0.1111	0.1111
0	0.2	0.0998	0.125	0.125
0	0.3	0.0994	0.1429	0.1429
0	0.4	0.0983	0.1667	0.1667
0	0.5	0.0962	0.2	0.2
0	0.6	0.0924	0.25	0.25
0.1	0	0.1111	0.1111	0.1111
0.1	0.1	0.1111	0.125	0.125
0.1	0.2	0.1109	0.1429	0.1429
0.1	0.3	0.1103	0.1667	0.1667
0.1	0.4	0.109	0.2	0.2
0.1	0.5	0.1064	0.25	0.25
0.2	0	0.125	0.125	0.125
0.2	0.1	0.125	0.1429	0.1429
0.2	0.2	0.1247	0.1667	0.1667
0.2	0.3	0.124	0.2	0.2
0.2	0.4	0.1224	0.25	0.25
0.3	0	0.1429	0.1429	0.1429
0.3	0.1	0.1428	0.1667	0.1667
0.3	0.2	0.1425	0.2	0.2
0.3	0.3	0.1416	0.25	0.25

Where

$\mathrm{var}\!\left(\theta^\wedge\right)p/\theta^2$ =Proposed variance from (3.5) $\mathrm{var}\!\left(\theta^\wedge\right)k/\theta^2$ = variance by Kambo from (5.5) $\mathrm{var}\!\left(\theta^\wedge\right)m\,\&\,n/\theta^2$ = variance by Mehrotra and Nanda from (5.2)

q1	q2	n=20 $\mathrm{var}\!\left(\theta^\wedge\right)p/\theta^2$	n=20 $\mathrm{var}\!\left(\theta^\wedge\right)k/\theta^2$	n=20 $\mathrm{var}\!\left(\theta^\wedge\right)m\,\&\,n/\theta^2$
0	0	0.05	0.05	0.05
0	0.1	0.05	0.0556	0.0556
0	0.2	0.0499	0.0625	0.0625
0	0.3	0.0497	0.0714	0.0714
0	0.4	0.0491	0.0833	0.0833
0	0.5	0.0481	0.1	0.1
0	0.6	0.0462	0.125	0.125
0.1	0	0.0556	0.0556	0.0556
0.1	0.1	0.0555	0.0625	0.0625
0.1	0.2	0.0555	0.0714	0.0714
0.1	0.3	0.0552	0.0833	0.0833
0.1	0.4	0.0545	0.1	0.1
0.1	0.5	0.0532	0.125	0.125
0.2	0	0.0625	0.0625	0.0625

q1	q2	n=20 $\mathrm{var}(\hat{\theta})p/\theta^2$	n=20 $\mathrm{var}(\hat{\theta})k/\theta^2$	n=20 $\mathrm{var}(\hat{\theta})m\,\&\,n/\theta^2$
0.2	0.1	0.0625	0.0714	0.0714
0.2	0.2	0.0624	0.0833	0.0833
0.2	0.3	0.062	0.1	0.1
0.2	0.4	0.0612	0.125	0.125
0.3	0	0.0714	0.0714	0.0714
0.3	0.1	0.0714	0.0833	0.0833
0.3	0.2	0.0713	0.1	0.1
0.3	0.3	0.0708	0.125	0.125

q1	q2	n=30 $\mathrm{var}(\hat{\theta})p/\theta^2$	n=30 $\mathrm{var}(\hat{\theta})k/\theta^2$	n=30 $\mathrm{var}(\hat{\theta})m\,\&\,n/\theta^2$
0	0	0.0333	0.0333	0.0333
0	0.1	0.0333	0.037	0.037
0	0.2	0.0333	0.0417	0.0417
0	0.3	0.0331	0.0476	0.0476
0	0.4	0.0328	0.0556	0.0556
0	0.5	0.0321	0.0667	0.0667
0	0.6	0.0308	0.0833	0.0833
0.1	0	0.037	0.037	0.037
0.1	0.1	0.037	0.0417	0.0417
0.1	0.2	0.037	0.0476	0.0476
0.1	0.3	0.0368	0.0556	0.0556
0.1	0.4	0.0363	0.0667	0.0667
0.1	0.5	0.0355	0.0833	0.0833
0.2	0	0.0417	0.0417	0.0417
0.2	0.1	0.0417	0.0476	0.0476
0.2	0.2	0.0416	0.0556	0.0556
0.2	0.3	0.0413	0.0667	0.0667
0.2	0.4	0.0408	0.0833	0.0833
0.3	0	0.0476	0.0476	0.0476
0.3	0.1	0.0476	0.0556	0.0556
0.3	0.2	0.0475	0.0667	0.0667
0.3	0.3	0.0472	0.0833	0.0833

q1	q2	n=50 $\mathrm{var}(\hat{\theta})p/\theta^2$	n=50 $\mathrm{var}(\hat{\theta})k/\theta^2$	n=50 $\mathrm{var}(\hat{\theta})m\,\&\,n/\theta^2$
0	0	0.02	0.02	0.02
0	0.1	0.02	0.0222	0.0222
0	0.2	0.02	0.025	0.025
0	0.3	0.0199	0.0286	0.0286
0	0.4	0.0197	0.0333	0.0333
0	0.5	0.0192	0.04	0.04
0	0.6	0.0185	0.05	0.05
0.1	0	0.0222	0.0222	0.0222
0.1	0.1	0.0222	0.025	0.025
0.1	0.2	0.0222	0.0286	0.0286
0.1	0.3	0.0221	0.0333	0.0333
0.1	0.4	0.0218	0.04	0.04
0.1	0.5	0.0213	0.05	0.05
0.2	0	0.025	0.025	0.025
0.2	0.1	0.025	0.0286	0.0286
0.2	0.2	0.0249	0.0333	0.0333
0.2	0.3	0.0248	0.04	0.04

q1	q2	n=50	n=50	n=50
		$\mathrm{var}\!\left(\theta^{\wedge}\right)p/\theta^2$	$\mathrm{var}\!\left(\theta^{\wedge}\right)k/\theta^2$	$\mathrm{var}\!\left(\theta^{\wedge}\right)m\,\&\,n/\theta^2$
0.2	0.4	0.0245	0.05	0.05
0.3	0	0.0286	0.0286	0.0286
0.3	0.1	0.0286	0.0333	0.0333
0.3	0.2	0.0285	0.04	0.04
0.3	0.3	0.0283	0.05	0.05

q1	q2	n=100	n=100	n=100
		$\mathrm{var}\!\left(\theta^{\wedge}\right)p/\theta^2$	$\mathrm{var}\!\left(\theta^{\wedge}\right)k/\theta^2$	$\mathrm{var}\!\left(\theta^{\wedge}\right)m\,\&\,n/\theta^2$
0	0	0.01	0.01	0.01
0	0.1	0.01	0.0111	0.0111
0	0.2	0.01	0.0125	0.0125
0	0.3	0.0099	0.0143	0.0143
0	0.4	0.0098	0.0167	0.0167
0	0.5	0.0096	0.02	0.02
0	0.6	0.0092	0.025	0.025
0.1	0	0.0111	0.0111	0.0111
0.1	0.1	0.0111	0.0125	0.0125
0.1	0.2	0.0111	0.0143	0.0143
0.1	0.3	0.011	0.0167	0.0167
0.1	0.4	0.0109	0.02	0.02
0.1	0.5	0.0106	0.025	0.025
0.2	0	0.0125	0.0125	0.0125
0.2	0.1	0.0125	0.0143	0.0143
0.2	0.2	0.0125	0.0167	0.0167
0.2	0.3	0.0124	0.02	0.02
0.2	0.4	0.0122	0.025	0.025
0.3	0	0.0143	0.0143	0.0143
0.3	0.1	0.0143	0.0167	0.0167
0.3	0.2	0.0143	0.02	0.02
0.3	0.3	0.0142	0.025	0.025

Tables 2. The MSE of MML estimates from doubly censored sample in term of θ^2 for n=10,20and 30

q1	q2	n=10	n=10	n=10
		$MSE\!\left(\theta^{\wedge}\right)p/\theta^2$	$MSE\!\left(\theta^{\wedge}\right)k/\theta^2$	$MSE\!\left(\theta^{\wedge}\right)m\,\&\,n/\theta^2$
0	0	0.0293	0.0493	0.0493
0	0.1	0.0292	0.0707	0.0707
0	0.2	0.0294	0.0989	0.0989
0	0.3	0.03	0.1371	0.1371
0	0.4	0.0317	0.1917	0.1917
0	0.5	0.0362	0.2747	0.2747
0	0.6	0.0473	0.4129	0.4129
0.1	0	0.0238	0.0485	0.0485
0.1	0.1	0.0238	0.0739	0.0739
0.1	0.2	0.0239	0.1087	0.1087
0.1	0.3	0.0246	0.1589	0.1589
0.1	0.4	0.0268	0.2361	0.2361
0.1	0.5	0.0323	0.3668	0.3668
0.2	0	0.0317	0.0629	0.0629
0.2	0.1	0.0316	0.0985	0.0985
0.2	0.2	0.0318	0.1507	0.1507
0.2	0.3	0.0327	0.2328	0.2328

q1	q2	n=10 $MSE(\hat{\theta})p/\theta^2$	n=10 $MSE(\hat{\theta})k/\theta^2$	n=10 $MSE(\hat{\theta})m\&n/\theta^2$
0.2	0.4	0.0355	0.3751	0.3751
0.3	0	0.0455	0.0863	0.0863
0.3	0.1	0.0454	0.1397	0.1397
0.3	0.2	0.0456	0.225	0.225
0.3	0.3	0.0468	0.3761	0.3761

Where

$MSE(\hat{\theta})p/\theta^2$ =Proposed MSE from (4.7) $MSE(\hat{\theta})k/\theta^2$ =by Kambo from (5.5.3) $MSE(\hat{\theta})m\&n/\theta^2$ =by Mehrotra and Nanda from (5.2.3)

q1	q2	n=20 $MSE(\hat{\theta})p/\theta^2$	n=20 $MSE(\hat{\theta})k/\theta^2$	n=20 $MSE(\hat{\theta})m\&n/\theta^2$
0	0	0.009	0.014	0.014
0	0.1	0.009	0.0225	0.0225
0	0.2	0.0091	0.0336	0.0336
0	0.3	0.0094	0.0484	0.0484
0	0.4	0.0104	0.0692	0.0692
0	0.5	0.0127	0.1003	0.1003
0	0.6	0.0186	0.1511	0.1511
0.1	0	0.0086	0.0148	0.0148
0.1	0.1	0.0086	0.0254	0.0254
0.1	0.2	0.0087	0.0397	0.0397
0.1	0.3	0.0091	0.0603	0.0603
0.1	0.4	0.0103	0.0916	0.0916
0.1	0.5	0.0132	0.1441	0.1441
0.2	0	0.014	0.0218	0.0218
0.2	0.1	0.014	0.0372	0.0372
0.2	0.2	0.0141	0.0596	0.0596
0.2	0.3	0.0147	0.0947	0.0947
0.2	0.4	0.0162	0.1556	0.1556
0.3	0	0.0219	0.0321	0.0321
0.3	0.1	0.0219	0.0555	0.0555
0.3	0.2	0.0221	0.0929	0.0929
0.3	0.3	0.0228	0.1591	0.1591

q1	q2	n=30 $MSE(\hat{\theta})p/\theta^2$	n=30 $MSE(\hat{\theta})k/\theta^2$	n=30 $MSE(\hat{\theta})m\&n/\theta^2$
0	0	0.0044	0.0067	0.0067
0	0.1	0.0044	0.0118	0.0118
0	0.2	0.0045	0.0184	0.0184
0	0.3	0.0047	0.0272	0.0272
0	0.4	0.0054	0.0394	0.0394
0	0.5	0.0071	0.0575	0.0575
0	0.6	0.0114	0.0865	0.0865
0.1	0	0.005	0.0078	0.0078
0.1	0.1	0.005	0.0144	0.0144
0.1	0.2	0.0051	0.0233	0.0233
0.1	0.3	0.0054	0.036	0.036
0.1	0.4	0.0062	0.0553	0.0553
0.1	0.5	0.0083	0.0875	0.0875
0.2	0	0.009	0.0124	0.0124

q1	q2	n=30 $MSE(\hat{\theta})p/\theta^2$	n=30 $MSE(\hat{\theta})k/\theta^2$	n=30 $MSE(\hat{\theta})m\&n/\theta^2$
0.2	0.1	0.009	0.0221	0.0221
0.2	0.2	0.0091	0.0363	0.0363
0.2	0.3	0.0095	0.0585	0.0585
0.2	0.4	0.0105	0.0968	0.0968
0.3	0	0.0144	0.019	0.019
0.3	0.1	0.0144	0.0338	0.0338
0.3	0.2	0.0146	0.0576	0.0576
0.3	0.3	0.0151	0.0997	0.0997

6. DISCUSSION AND CONCLUSION

1) The purpose of conducting empirical study is to see the closeness of MML estimators to ML estimator of Kambo .Since it has been proved that MML estimator of Mehrotra and Nanda are exactly same to Kambo estimator and they also equivalent in terms of asymptotic variance and MSE.

2) The asymptotic variance of Kambo, Mehrotra & Nanda and Proposed asymptotic variance are given in the tables (5.1) for n= 10 ,20 ,30 ,50 and 100.It can be observed that Proposed asymptotic variance is minimum as compared to the asymptotic variance of Kambo and Mehrotra & Nanda .The asymptotic variance of Kambo and Mehrotra & Nanda are exactly same.

3) For $q_1=q_2=0$. no censoring scheme is involved in other words there is no missing element in the sample then the Proposed asymptotic variance ,asymptotic variance by Kambo and asymptotic variance by Mehrotra and Nanda are exactly same for any sample of size n.

4) The Mean square error of Kambo, Mehrotra & Nanda and Proposed MSE are given in the tables (5.2) for n=10 ,20 and 30.The MSE of scale parameter by Mehrotra & Nanda and ML estimator by Kambo are exactly same. It also observed that Proposed MSE are minimum as compared to the MSE's of Kambo and Mehrotra & Nanda.

5) As the sample size increases the Mean square errors and asymptotic variances of all MML and ML estimators tends to decreases .

6) Proposed Mean square error and asymptotic variance is minimum as compared to the Mean square error and asymptotic variance of Kambo and Mehrotra & Nanda respectively.

For further the authors refer [8-133], to recent advances in statistics for the readers.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Aleem M, Jamal F. Modified Inference about the scale parameter of the Inverse Weibule Distribution by using type ll censored sample", Submitted; 2008.
2. Mehrotra KG, Nanda P . Unbiased estimator of parameter by order statistics in the case of censored samples . Biometrika. 1974;61:601-606.
3. Kambo NS. Maximum likelihood estimators of the location and scale parameters of the exponential distribution from a censored sample, comm.. Statist. Theory methods A. 1978;12:1129-1132.
4. Suresh RP. Estimation of location and scale parameters in a two parameter exponential distribution from a censored sample , statistical methods . 2004;6(1):82-89.
5. John TT, Chen P. Selection of the Best using Censored Data from (log-) location - Scale Populations. Preprint submitted to journal of Statistical Planning and Inference; 2003.
6. Aleem M. Classical properties, ratio, product and inverse moments of order statistics and characterization from inverse weibull distribution . J. Of stats. 2004;Xi(1):17- 24.
7. Abu-Dayyeh W, Al-Saw-E. Modified Infrence about scale parameter of logstic distribution using type II cendored sample Pak. J. Stat. 2006;22(2):157-160.
8. Aleem M, Jamal F. Modified Inference About the Scale Parameter of The Inverse Weibull Distribution By Using Type II Censored Sample. Sindh University Research Journal-Surj (Science Series). 2009;41(1).

9. Aleem M, Jamal F, Rafique L. Modified Inference about the Scale Parameter of the Weibull Distribution By Using Type II Censored Sample, Proc. 5th International Statistical Conference. 2009;17:203-210.

10. Aleem M, Jamal F, Shah MAA. An Empirical Study on Modified Maximum Likelihood Estimator and Maximum Likelihood Estimator for Inverse Weibull Distribution by Using Type-II Censored Sample. Proc. 11th International Conference on Statistical Sciences, DG Khan, Pakistan. 2013;25:81-92.

11. Mohsin M, Safdar S, Asghar F, Jamal F. Assessment of drinking water quality and its impact on residents health in Bahawalpur city. International Journal of Humanities and Social Science. 2013;3(15):114-128.

12. Jamal F, Nasir MA. An empirical study on modified ML estimator and ML estimator of Weibull distribution by using type II censored sample. Journal of Engineering Research and Applications . 2014;4(1):13-20.

13. Mohsin M, Jamal F, Khan AA, Ajmal F. Transformation of fertile agricultural soil into housing schemes: A case of Bahawalpur City, Punjab, Pakistan. International Review of Social Sciences and Humanities. 2014;6(2):141-156.

14. Mohsin M, Jamal F, Ajmal F. Impact of Mango Orchard Diseases on Growers Economic Life in Ahmedpur East, Bahawalpur, Pakistan. Academic Research International. 2014;5(2):196-204.

15. Jamal F, Nasir MA, Nasir JA. A mixture of modified inverse Weibull distribution. Journal of Statistics Applications and Probability Letters. 2014;2:31-46.

16. Shah MAA, Aleem M, Arshed N. Statistical analysis of the factors affecting inflation in Pakistan. Middle-East Journal of Scientific Research. 2014;21(1):181-189.

17. Mohsin M, Jamal F, Khan A. A. Historical evolution and current legal scenario of housing schemes in Bahawalpur city, Pakistan: A review and comments. International Journal of Innovation and Applied Studies. 2014;7(2):748-756.

18. Jamal F, Tahir MH, Alizadeh M, Nasir MA. On Marshall-Olkin Burr X family of distribution. Tbilisi Mathematical Journal. 2017;10(4):175-199.

19. Mohsin M, Anwar MM, Jamal F, Ajmal F, Breuste J. Assessing the role and effectiveness of kitchen gardening toward food security in Punjab, Pakistan: a case of district Bahawalpur. International Journal of Urban Sustainable Development. 2017;9(1):64-78.

20. Nasir MA, Tahir MH, Jamal F, Ozel G. A new generalized Burr family of distributions for the lifetime data. Journal of Statistics Applications and Probability. 2017;6(2):401-417.

21. Jamal F, Nasir MA, Tahir MH, Montazeri NH. The odd Burr-III family of distributions. Journal of Statistics Applications and Probability. 2017;6(1):105-122.

22. Nasir MA, Aljarrah M, Jamal F, Tahir MH. A new generalized Burr family of distributions based on quantile function. Journal of Statistics Applications and Probability. 2017;6(3):1-14.

23. Chesneau C, Jamal F. On a new special member of the Weibull-X family of distributions. Eurasian Bulletin of Mathematics. 2018;1(3):94-97.

24. Jamal F, Nasir MA. Generalized Burr X family of distribution, International Journal of Mathematics and Statistics. 2018;19(1):55-73.

25. Arshad RMI, Jamal F, Shah MAA, Ozel G, Imran S. Stringency Criterion for Modality Tests. Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences. 2018;55(3):77-83.

26. Nasir A, Bakouch HS, Jamal F. Kumaraswamy odd Burr G family of distributions with applications to reliability data. Studia Scientiarum Mathematicarum Hungarica. 2018;55(1):94-114.

27. Reyad H, Jamal F, Othman S, Hamedani GG. The transmuted Gompertz-G family of distributions: properties and applications. Tbilisi Mathematical Journal. 2018;11(3):47-67.

28. Jamal F, Aljarrah MA, Tahir MH, Nasir MA. A new extended generalized Burr-III family of distributions. Tbilisi Mathematical Journal. 2018;11(1):59-78.

29. Reyad H, Alizadeh M, Jamal F, Othman S. The Topp Leone odd Lindley-G family of distributions: properties and applications. Journal of Statistics and Management Systems. 2018;21(7):1273-1297.

30. Elgarhy M, Arslan Nasir M, Jamal F, Ozel G. The type II Topp-Leone generated family of distributions: Properties and applications. Journal of Statistics and Management Systems. 2018;21(8): 1529-1551.

31. Nasir MA, Ozel G, Jamal F. The Burr XII Uniform Distribution: Theory And Applications. Journal of Reliability and Statistical Studies. 2018;11(2):143-158.

32. Nasir MA, Jamal F, Silva GO, Tahir MH. Odd Burr-G Poisson family of distributions. Journal of Statistics Applications and Probability. 2018;7 (1):9-28.

33. Reyad H, Jamal F, Othman S, Hamedani GG. The transmuted odd Lindley-G family of distributions. Asian Journal of Probability and Statistics. 2018;1(3):1-25.

34. Nasir MA, Korkmaz MC, Jamal F, Yousof HM. On a new Weibull Burr XII distribution for lifetime data. Sohag Journal of Mathematics. 2018;5(2):47-56.

35. Ahmad Z, Ampadu CB, Hamedani GG, Jamal F, Nasir MA. The New Exponentiated TX Class of Distributions: Properties, Characterizations and Application. Pakistan Journal of Statistics and Operation Research. 2019;941-962.

36. Bantan RA, Jamal F, Chesneau C, Elgarhy M. Truncated inverted Kumaraswamy generated family of distributions with applications. Entropy. 2019;21(11):1089.

37. Marzouk W, Jamal F, Ahmed A, Hadi N, Ahmed A. The Generalized Odd Lomax Generated Family of Distributions with Applications. Gazi University Journal of Science. 2019;32(2): 737-755.

38. Nasir A, Yousof HM, Jamal F, Korkmaz MÇ. The exponentiated Burr XII power series distribution: properties and applications. Stats. 2019;2(1):15-31.

39. Reyad HM, Alizadeh M, Jamal F, Othman S, Hamedani GG. The exponentiated generalized Topp Leone-G family of distributions: Properties and applications. Pakistan Journal of Statistics and Operation Research. 2019;1-24.

40. Reyad H, Jamal F, Othman S, Yahia N. The Topp Leone Generalized Inverted Kumaraswamy Distribution: Properties and Applications. Asian Research Journal of Mathematics. 2019;13(3):1-15.

41. Nasir A, Jamal F. Some new results and applications on the min WB distribution. Eurasian Bulletin of Mathematics. 2019;2(2):36-47.

42. Farrukh Jamal, Elbatal, Ibrahim, Christophe Chesneau, Mohammed Elgarhy, and Amal Hassan.). Modified Beta Generalized Linear Failure Rate Distribution: Theory and Applications. Journal of Prime Research in Mathematics. 2019;15:21-48.

43. Jamal F, Nasir M. Some new members of the TX family of distributions, Proc. 5th International Statistical Conference. 2019;33:113-120.

44. Jamal F, Reyad HM, Ahmed SO, Shah MAA, Altun E. Exponentiated Half-Logistic Lomax Distribution ith Properties And Application. NED University Journal of Research. 2019;16(2):1-11.

45. Arshad RMI, Imran S, Jamal F, Ozel G. A Comparison of Modality Tests Based on Real Life Data Applications. Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences. 2019;56(1):69-75.

46. Jamal F, Reyad H, Chesneau C, Nasir MA, Othman S. The Marshall-Olkin odd Lindley-G family of distributions: Theory and applications. Punjab University Journal of Mathematics. 2019;51(7):111-125.

47. Elbatal I, Jamal F, Chesneau C, Elgarhy M, Alrajhi S. The Modified Beta Gompertz Distribution: Theory and Applications. Mathematics. 2019;7(1):3.

48. Arshad RMI, Chesneau C, Jamal F. The odd gamma Weibull-geometric model: theory and applications. Mathematics. 2019;7(5):399.

49. Jamal F, Chesneau C. A new family of polyno-expo-trigonometric distributions with applications. Infinite Dimensional Analysis, Quantum Probability and Related Topics. 2019;22(4):1950027.

50. Bantan RA, Jamal F, Chesneau C, Elgarhy M. A New Power Topp–Leone Generated Family of Distributions with Applications. Entropy. 2019;21(12):1177.

51. ZeinEldin RA, Chesneau C, Jamal F, Elgarhy M. Different Estimation Methods for Type I Half-Logistic Topp–Leone Distribution. Mathematics. 2019;7(10):985.

52. Jamal F, Arslan Nasir M, Ozel G, Elgarhy M, Mamode Khan N. Generalized inverted Kumaraswamy generated family of distributions: theory and applications. Journal of Applied Statistics. 2019;46(16):2927-2944.

53. ZeinEldin RA, Chesneau C, Jamal F, Elgarhy M. Statistical Properties and Different Methods of Estimation for Type I Half Logistic Inverted Kumaraswamy Distribution. Mathematics. 2019;7(10): 1002.

54. Bantan RA, Jamal F, Chesneau C, Elgarhy M. Truncated inverted Kumaraswamy generated family of distributions with applications. Entropy. 2019;21(11);1089.

55. ZeinEldin RA, Jamal F, Chesneau C, Elgarhy M. Type II Topp–Leone Inverted Kumaraswamy Distribution with Statistical Inference and Applications. Symmetry. 2019;11(12):1459.

56. Mohsin M, Anwar MM, Dawood M, Shah MMA, Jamal F, Basit M. Impact of land use change on land values: A case of Jhangiwala, Bahawalpur City, Pakistan, Journal of Biodiversity and Environmental Sciences. 2019;14(6);152-160.

57. Marzouk W, Jamal F, Ahmed A, Hadi N, Ahmed A. The Generalized Odd Lomax Generated Family of Distributions with Applications. Gazi University Journal of Science. 2019;32(2).

58. Shah MAA, Özel G, Jamal F, Awan ST. The Social and Economic Problems of Home-based and Outdoor Working Women in Punjab, Pakistan. E-Journal of International and Comparative Labour Studies. 2019;8(2):98-108.

a. Zubair M, Ali A, Naeem S, Jamal, F Chesneau C. Emotion Recognition from Facial, Expression Using Machine Vision Approach, Journal of Applied and Emerging Sciences. 2020;10(1):35-40.

59. Jamal F, Ozel G, Arshad IMR, Imran S. A comparison of non parametric modality tests. A physical and computational of sciences. 2020;57(1):17-20.

60. Alizadeh M, Jamal F, Yousof HM, Khanahmadi M, Hamedani GG. Flexible Weibull Generated Family of Distributions: Characterizations, Mathematical Properties and Applications. University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics. 2020;82(1): 145-150.

61. Aldahlan MA, Jamal F, Chesneau C, Elbatal I, Elgarhy M. Exponentiated power generalized Weibull power series family of distributions: Properties, estimation and applications. Plos one. 2020;15(3): e0230004.

62. ZeinEldin RA, Chesneau C, Jamal F, Elgarhy M. Different Estimation Methods for Type I Half-Logistic Topp–Leone Distribution. Mathematics. 2019;7(10):985.

63. Jamal F, Chesneau C, Nasir MA, Saboor A, Altun E, Khan MA. On a modified Burr XII distribution having flexible hazard rate shapes. Mathematica Slovaca. 2020;70(1):193-212.

64. Shah MAA, Ozel G, Chesneau C, Mohsin M, Jamal F, Bhatti MF. A statistical study of the determinants of rice crop production in Pakistan. Pakistan Journal of Agricultural Research. 2020;33(1):97.

65. Bantan RA, Jamal F, Chesneau C, Elgarhy M. Type II Power Topp-Leone Generated Family of Distributions with Statistical Inference and Applications. Symmetry. 2020;12(1):75.

66. Al-Marzouki S, Jamal F, Chesneau C, Elgarhy M. Type II Topp Leone Power Lomax Distribution with Applications. Mathematics. 2020;8(1):4.

67. Jamal F, Chesneau C, Elgarhy M. Type II general inverse exponential family of distributions. Journal of Statistics and Management Systems. 2020;23(3):617-641.

68. Chesneau C, Jamal F. On a new special member of the Weibull-X family of distributions. Eurasian Bulletin of Mathematics. 2020;(ISSN: 2687-5632),3(2):56-72.

69. Shah MAA, Mashwani WK, Kumam W, Kumam P, Chesneau C, Jamal F, Khan HU. Application of Mixed Sampling to Real Life Data: A Case Study on Socio-Economic Determinants by Using SEM and CFA Techniques. Mathematics. 2020;8(3):337.

70. Bantan RA, Zeineldin RA, Jamal F, Chesneau C. Determination of the Factors Affecting King Abdul Aziz University Published Articles in ISI by Multilayer Perceptron Artificial Neural Network. Mathematics. 2020;8(5):766.

71. Al-Marzouki S, Jamal F, Chesneau C, Elgarhy M. Erratum: Al-Marzouki S, et al. Type II Topp Leone Power Lomax Distribution with Applications. Mathematics 2020, 8, 4. Mathematics. 2020;8(6):871.

72. Bantan RA, Elgarhy M, Chesneau C, Jamal F. Estimation of Entropy for Inverse Lomax Distribution under Multiple Censored Data. Entropy. 2020;22(6):601.

73. Ali A, Qadri S, Mashwani WK, Brahim Belhaouari S, Naeem S, Rafique S, Anam S. Machine learning approach for the classification of corn seed using hybrid features. International Journal of Food Properties. 2020;23(1):1110-1124.

74. Ali A, Qadri S, Khan Mashwani W, Kumam W, Kumam P, Naeem S, Sulaiman M. Machine Learning Based Automated Segmentation and Hybrid Feature Analysis for Diabetic Retinopathy Classification Using Fundus Image. Entropy. 2020;22(5):567.

75. Naeem S, Ali A, Qadri S, Mashwani WK, Tairan N, Shah H, Anam S. Machine-Learning Based Hybrid-Feature Analysis for Liver Cancer Classification Using Fused (MR and CT) Images. Applied Sciences. 2020;10(9):3134.

76. Jamal F, Reyad HM, Ahmed SO, Ali Shah SMA. Mathematical Properties and Applications of Minimum Gumbel Burr

Distribution. NED University Journal of Research. 2020;17(2);1-14.

77. Jamal F, Bakouch HS, Nasir MA. Odd Burr III G-Negative Binomial Family with Application. Journal of Testing and Evaluation. 2020;49(5):1-21.

78. Handique L, Jamal F, Chakraborty S. On a family that unifies Generalized Marshall-Olkin and Poisson-G family of distribution. Accepted Manuscript, arXiv preprint ar.2020;Xiv:2006.05816.

79. Bantan RA, Jamal F, Chesneau C, Elgarhy M. On a new result on the ratio exponentiated general family of distributions with applications. Mathematics. 2020;8(4): 598.

80. Bantan RA, Chesneau C, Jamal F, Elgarhy M. On the Analysis of New COVID-19 Cases in Pakistan Using an Exponentiated Version of the M Family of Distributions. Mathematics. 2020;8(6):953.

81. Almarashi AM, Badr MM, Elgarhy M, Jamal F, Chesneau C. Statistical Inference of the Half-Logistic Inverse Rayleigh Distribution. Entropy. 2020;22(4):449.

82. Almarashi AM, Elgarhy M, Jamal F, Chesneau C. The Exponentiated Truncated Inverse Weibull-Generated Family of Distributions with Applications. Symmetry. 2020;12(4):650.

83. Chesneau C, Tomy L, Gillariose J, Jamal F. The inverted modified Lindley distribution. Journal of Statistical Theory and Practice. 2020;14(3):1-17.

84. Chakraborty S, Handique L, Jamal F. The Kumaraswamy Poisson-G family of distribution: its properties and applications. Annals of Data Science. 2020;7(4):1-19.

85. Nasir MA, Tahir MH, Chesneau C, Jamal F, Shah MAA. The Odds Generalized Gamma-G Family of Distributions: Properties, Regressions and Applications. Statistica. 2020;80(1):3-38.

86. Reyad H, Jamal F, Özel G, Othman S. The Poisson exponential-G family of distributions with properties and applications. Journal of Statistics and Management Systems. 2020;23(7):1-24.

87. Badr MM, Elbatal I, Jamal F, Chesneau C, Elgarhy M. The Transmuted Odd Fréchet-G Family of Distributions: Theory and Applications. Mathematics. 2020;8(6):958.

88. Aldahlan MA, Jamal F, Chesneau C, Elgarhy M, Elbatal I. The truncated Cauchy power family of distributions with inference and applications. Entropy. 2020;22(3):346.

89. Al-Marzouki S, Jamal F, Chesneau C, Elgarhy M. Topp-Leone Odd Fréchet Generated Family of Distributions with Applications to COVID-19 Data Sets. Computer Modeling in Engineering & Sciences. 2020;125(1):437-458.

90. Bantan RA, Jamal F, Chesneau C, Elgarhy M. Type II Power Topp-Leone Generated Family of Distributions with Statistical Inference and Applications. Symmetry. 2020;12(1):75.

91. Jamal F, Chesneau C, Elgarhy M. Type II general inverse exponential family of distributions. Journal of Statistics and Management Systems. 2020;23(3):617-641.

92. Al-Babtain AA, Elbatal I, Chesneau C, Jamal F. Box-Cox Gamma-G Family of Distributions: Theory and Applications. Mathematics. 2020;8(10):1801.

93. Bantan RA, Chesneau C. Jamal F, Elgarhy M, Tahir MH, Ali A, Anam,S. Some New Facts about the Unit-Rayleigh Distribution with Applications. Mathematics. 2020;8(11):1954.

94. Gillariose J, Tomy L, Jamal F, Chesneau C. The Marshall-Olkin Modified Lindley Distribution: Properties and Applications. Journal of Reliability and Statistical Studies. 2020;13(1):177-198.

95. Al-Babtain AA, Elbatal I, Chesneau C, Jamal F. The Transmuted Muth Generated Class of Distributions with Applications. Symmetry. 2020;12(10):1677.

96. Bantan RA, Ali A, Naeem S, Jamal F, Elgarhy M, Chesneau C. Discrimination of sunflower seeds using multispectral and texture dataset in combination with region selection and supervised classification methods. Chaos: An Interdisciplinary Journal of Nonlinear Science. 2020;30(11):113142.

97. Marzouk W, Handique L, Ahmed, NHA, Jamal F, Rahman AAA. The Inverted Modifed Family of Distributions with Applications to Reliability Theory. Journal of Modern Applied Statistical Methods, In Press; 2021.

98. Reyad MH, Afify ZA, Jamal F, Othman S The Extended Power Lindley-G Family of Distributions: Properties and Applications. Journal of Modern Applied Statistical Methods, In Press; 2021.

99. Jamal F, Bakouch H, Nasir A. A truncated general-G class of distributions with application to truncated Burr-G family, REVSTAT - Statistical Journal, In Press; 2021.

100. Jamal F, Chesneau C, Aidi. K. The Sine Extended Odd Fréchet-G Family of Distribution with Applications to Complete and Censored Data, Mathematica Slovaca. In Press; 2021.

101. Jamal F, Reyad HM, Nasir MA, Chesneau C, M. Akbar Ali Shah. MAA, Ahmed SO. The Topp Leone Weibull-Lomax Distributions: Properties, Regression Model and Applications, NED University Journal of Research-Applied Sciences, In Press; 2021.

102. Jamal F, Chesneau C. The sine Kumaraswamy-G family of distributions. Journal of Mathematical Extension. In Press; 2021.

103. Handique L, Shah MAA, Mohsin M, Jamal F. Properties and Applications of a New Member of The T-X Family of Distributions. Thailand Statistician. In Press; 2021.

104. Handique L, Chakraborty S, Jamal F. Beta Poisson-G Family of Distributions: Its Properties and Application with Failure Time Data. Thailand Statistician. In Pres; 2021.

105. Marzouk W, Handique L, A-Hadi NA, Jamal F, Rahman AAA. The Generalized Odd Linear Exponential Family of Distributions with Applications to Reliability Theory Journal of Modern Applied Statistical Methods. In Press; 2021.

106. Reyad H, Jamal F, Hamedani GG, Othman S. The Alpha Power Transformrd Dagum Distribution: Properties and Applications. Journal of Data Science. In press; 2021.

107. Hassan S, Bakouch HS, Jamal F, Chesneau C, Nasir A. A new transmuted family of distributions: Properties and estimation with applications. Calcutta Statistical Association Bulletin, In press; 2021.

108. Arshad RMI, Tahir MH, Chesneau C, Jamal F. The Gamma Kumaraswamy-G family of distributions: Theory, inference and applications. Statistics in Transition New Series. 2020;21(5):17–40

109. Ali A, Mashwani WK, Tahir MH, Brahim Belhaouari S, Alrabaiah H, Naeem S, Nasir J, Jamal F, Chesneau C. Statistical features analysis and discrimination of maize seeds utilizing machine vision approach. Journal of Intelligent & Fuzzy Systems. 2020;1-12.

110. Ali A, Nasir JA, Ahmed MM, Naeem S, Anam S, Jamal F, Chesneau C, Zubair M, Anees MS. Machine Learning Based Statistical Analysis of Emotion Recognition using Facial Expression. RADS Journal of Biological Research and Applied Science. 2020;11(1):1-8.

111. Aqib, Ali, Wali Khan, Naeem Mashwani, Uddin Samreen, Muhammad Irfan, Wiyada Kumam, Poom Kumam, Hussam Alrabaiah, Farrukh Jamal, and Christophe Chesneau. "COVID-19 Infected Lung Computed Tomography Segmentation and Supervised Classification Approach." Computers, Materials, & Continua. 2021;391-407.

112. Naeem, Samreen, Wali Khan Mashwani, Aqib Ali, M. Irfan Uddin, Marwan Mahmoud, Farrukh Jamal, and Christophe Chesneau. "Machine Learning-based USD/PKR Exchange Rate Forecasting Using Sentiment Analysis of Twitter Data." CMC-Computers Materials & Continua. 2021;67(3):3451-3461.

113. Jamal F, Chesneau C, Aidi K, Ali A. Theory and Application of the Power Ailamujia Distribution, Journal of Mathematical Modeling. 2021;9(3):391-413.

114. Afify AZ, Cordeiro GM, Ibrahim NA, Jamal F, Elgarhy M, Nasir MA. The Marshall-Olkin Odd Burr III-G Family: Theory, Estimation, and Engineering Applications. IEEE Access. 2021;9:4376-4387.

115. Abdulhakim A. Al-Babtain, Rehan AK Sherwani, Ahmed Z. Afify, Khaoula Aidi, M. Arslan Nasir, Farrukh Jamal, Abdus Saboor. The extended Burr-R class: properties, applications and modified test for censored data. 2021;6(3):2912-2931.

116. Ramadan A. ZeinEldin, Christophe Chesneau, Farrukh Jamal, Mohammed Elgarhy , Abdullah M. Almarashi and Sanaa Al-Marzouki. Generalized Truncated Frechet Generated Family Distributions and ´ Their Applications. 2021;126(2):791-819.

117. Farrukh Jamal, Christophe Chesneau, Abdus Saboor, Muhammad Aslam, M.H.Tahir and Wali Khan Mashwani. The U family of distributions: Properties and Applications, Accepted in Mathematica Slovaca. 2021;00(00):00.

118. Naeem S, Ali A, Chesneau C, Tahir MH, Jamal F, Sherwani RAK, Ul Hassan M. The Classification of Medicinal Plant Leaves Based on Multispectral and Texture Feature Using Machine Learning Approach. Agronomy. 2021;11:263.

119. Al-Marzouki S, Jamal F, Chesneau C, Elgarhy M. Half Logistic Inverse Lomax Distribution with Applications. Symmetry. 2021;13(2): 309.
Available:https://doi.org/10.3390/sym1302030 9

120. Al-Marzouki, Sanaa, Christophe Chesneau, Sohail Akhtar, Jamal Abdul Nasir, Sohaib Ahmad, Sardar Hussain, Farrukh Jamal, Mohammed Elgarhy, and M. El-Morshedy. "Estimation of finite population mean under PPS in presence of maximum and minimum

values." AIMS Mathematics. 2021;6(5):397-5409.

121. Bantan, Rashad, Amal S. Hassan, Ehab Almetwally, M. Elgarhy, Farrukh Jamal, Christophe Chesneau, and Mahmoud Elsehetry. "Bayesian Analysis in Partially Accelerated Life Tests for Weighted Lomax Distribution." CMC-Computers Materials & Continua. 2021;68(3):2859-2875.

122. Khan Sherwani, Muhammad Waqas, Nadia Saeed, Muhammad Farooq, Muhammad Ali Raza, Farrukh, Transmuted Inverted Kumaraswamy Distribution: Theory and Applications Rehan Ahmad" Jamal Punjab University Journal of Mathematics. 2021;53(3):29-45.

123. Aqib, Ali, Wali Khan, Naeem Mashwani, Uddin Samreen, Muhammad Irfan, Wiyada Kumam, Poom Kumam, Hussam Alrabaiah, Farrukh Jamal, and Christophe Chesneau. "COVID-19 Infected Lung Computed Tomography Segmentation and Supervised Classification Approach." Computers, Materials, & Continua. 2021;391-407.

124. Farooq M, Sarfraz S, Chesneau C, Ul Hassan M, Raza MA, Sherwani RAK, Jamal F. Computing Expectiles Using k-Nearest Neighbours Approach. Symmetry. 2021;13: 645.
    Available:https://doi.org/10.3390/sym1304064 5

125. Abdullah Ali H. Ahmadini, Wali Khan Mashwani, Rehman Ahmad Khan Sherwani, Shokrya S. Alshqaq, Farrukh Jamal, Miftahuddin Miftahuddin, Kamran Abbas, Faiza Razaq, Mohammed Elgarhy, and Sanaa Al-Marzouki" Estimation of Constant Stress Partially Accelerated Life Test for Fréchet Distribution with Type-I Censoring". Mathematical Problems in Engineering. 2021;Article ID 9957944,8.

126. Jamal F, Chesneau C, Bouali DL, Ul Hassan M. Beyond the Sin-G family: The transformed Sin-G family. PLoS ONE. 2021;16(5): e0250790.

Available:https://doi.org/10.1371/journal.pone. 0250790

127. Bantan R, Elsehetry M, Hassan AS, Elgarhy M, Sharma D, Chesneau C, Jamal F. A Two-Parameter Model: Properties and Estimation under Ranked Sampling. Mathematics. 2021;9:1214.
    Available:https://doi.org/10.3390/math911121 4

128. Almarashi AM, Khan K, Chesneau C, Jamal F. Group Acceptance Sampling Plan Using Marshall–Olkin Kumaraswamy Exponential (MOKw- E) Distribution. Processes. 2021;9:1066.
    Available:https:// doi.org/10.3390/pr9061066.

129. Muhammad Mohsin , Shafaqat Mehmood, Farrukh Jamal and Muhammad Mushahid Anwer Modeling and Forecasting of Declining Area and Production of Mango Orchards in District Bahawalpur, Pakistan: A Case Of Tehsil Ahmedpur East J Agric. Res. 2021;59(2):197-212.

130. Muhammad Rasheed, Khadija Noreen, Rashid Ahmed, MH. Tahir & Farrukh Jamal. Some useful classes of minimal weakly balanced neighbor designs in circular blocks of two different sizes, Communications in Statistics - Theory and Methods; 2021.
    DOI: 10.1080/03610926.2021.1975135

131. Bantan, Rashad AR, Christophe Chesneau, Farrukh Jamal, Ibrahim Elbatal, and Mohammed Elgarhy. The Truncated Burr XG Family of Distributions: Properties and Applications to Actuarial and Financial Data. Entropy. 2021;23(8):1088.

132. Algarni, Ali, Abdullah M. Almarashi, Farrukh Jamal, Christophe Chesneau, and Mohammed Elgarhy. Truncated Inverse Lomax Generated Family of Distributions with Applications to Biomedical Data. Journal of Medical Imaging and Health Informatics. 2021;11(9):2425-2439.

133. Bantan, Rashad AR, Farrukh Jamal, Christophe Chesneau, and Mohammed Elgarhy. "Theory and Applications of the Unit Gamma/Gompertz Distribution. Mathematics. 2021;9(16):1850.