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On Cyclic Orthogonal Double Covers of Circulant Graphs using Infinite Graph Classes

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Research Article

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Abstract

An orthogonal double cover (ODC) of a graph H is a collection $\mathcal{G} = \{G_v : v \in V(H)\}$ of $|V(H)|$ subgraphs of H such that every edge of H is contained in exactly two members of $\mathcal{G}$ and for any two members G_u and G_v in $\mathcal{G}$, $|E(G_u) \cap E(G_v)|$ is 1 if $\{u,v\} \in E(H)$ and it is 0 if $\{u,v\} \notin E(H)$. An ODC $\mathcal{G}$ of H is *cyclic* (CODC) if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$. In this paper, the CODCs of certain circulants with a specific regularity by certain infinite graph classes are concerned.

Keywords: *Graph decomposition, orthogonal double covers, orthogonal labelling, circulants.*

1 Introduction

Let H be any graph and let $\mathcal{G} = \{G_1, G_2, \ldots, G_{|V(H)|}\}$ be a collection of $|V(H)|$ subgraphs of H. $\mathcal{G}$ is a *double cover* (DC) of H if every edge of H is contained in exactly two members in $\mathcal{G}$. If $G_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, for some graph G, then $\mathcal{G}$ is a DC of H by G. If $\mathcal{G}$ is a DC of H by G then $|V(H)||E(G)| = 2|E(H)|$.

A DC $\mathcal{G}$ of H is an *orthogonal double cover* (ODC) of H if there exists a bijective mapping $\phi : V(H) \to \mathcal{G}$ such that for every choice of distinct vertices u and v in $V(H)$,

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$$\left|E\left(\phi(u)\right)\cap E\left(\phi(v)\right)\right|=\begin{cases}1 \ if \ \{u,v\}\in E(H),\\ 0 \ if \ \{u,v\}\notin E(H).\end{cases}$$

If $G_i \cong G$ for all $i \in \{1,2,\ldots,|V(H)|\}$, then $\mathcal{G}$ is an ODC of H by G.

An *automorphism* of an ODC $\mathcal{G}=\{G_1,G_2,\ldots,G_{|V(H)|}\}$ of H is a permutation $\sigma: V(H)\to V(H)$ such that $\{\sigma(G_1),\sigma(G_2),\ldots,\sigma(G_{|V(H)|})\}=\mathcal{G}$, where for $i\in\{1,2,\ldots,|V(H)|\}$, $\sigma(G_i)$ is a subgraph of H with $V(\sigma(G_i))=\{\sigma(v): v\in V(G_i)\}$ and $E(\sigma(G_i))=\{\{\sigma(u),\sigma(v)\}:\{u,v\}\in E(G_i)\}$. An ODC $\mathcal{G}$ of H is *cyclic* (CODC) if the cyclic group of order |V(H)| is a subgroup of the automorphism group of $\mathcal{G}$, the set of all automorphisms of $\mathcal{G}$.

Throughout this article, we used the usual notation: K_n for the complete graph on n vertices, $K_{m,n}$ for the complete bipartite graph with partition sets of sizes m and n, P_{n+1} for a path on $n+1$ vertices, C_n for the cycle on n vertices, S_n for the star on $n+1$ vertices.

For a sequence $\{d_1,d_2,\ldots,d_k\}$ of positive integers with $1\le d_1\le d_2\le\ldots\le d_k\le\left\lfloor\frac{n}{2}\right\rfloor$, the circulant graph $Circ(n;\{d_1,d_2,\ldots,d_k\})$ has vertex set $\mathbb{Z}_n=\{0,1,\ldots,n\text{-}1\}$; two vertices v_1 and v_2 are adjacent if and only if $v_1-v_2\equiv\pm d_i \pmod n$ for some i, $i\in\{1,2,\ldots,k\}$. For an edge $\{v_1,v_2\}$ in $Circ(n;\{d_1,d_2,\ldots,d_k\})$, the length of $\{v_1,v_2\}$ is $\min\{|v_1-v_2|,\ n-|v_1-v_2|\}$.

Given two edges $e_1=\{v_1,v_2\}$ and $e_2=\{u_1,u_2\}$ of the same length l in $Circ(n;\{d_1,d_2,\ldots,d_k\})$, the *rotation-distance* $r(l)$ between e_1 and e_2 is $r(l)=\min\{r_1,\ r_2:\{v_1+r_1,v_2+r_1\}=e_2,\ \{u_1+r_2,u_2+r_2\}=e_1\}$, where addition and difference are calculated inside $\mathbb{Z}_n$ (that is, addition and difference are reduced modulo n). Note that if $r(l)=l$, then the edges e_1 and e_2 are adjacent; if $r(l)\neq l$, then the edges e_1 and e_2 are nonadjacent.

Consider the complete graph $K_n = Circ(n;\left\{1,2,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\right\})$. The author of [1] introduced the notion of an orthogonal labelling. Given a graph $G = (V,E)$ with $n-1$ edges, a $1-1$ mapping $\psi : V \to \mathbb{Z}_n$ is an *orthogonal labelling* of G if:

(1) For every $l \in \left\{1,2,\ldots,\left\lfloor \frac{(n-1)}{2} \right\rfloor\right\}$, G contains exactly two edges of length l, and exactly one edge of length $(n/2)$ if n is even, and

(2) $\left\{r(l) : \left\{l \in 1,2,\ldots,\left\lfloor \frac{n-1}{2} \right\rfloor\right\}\right\} = \left\{1,2,\ldots,\left\lfloor \frac{n-1}{2} \right\rfloor\right\}$.

The following theorem of Gronau et al. [1] relates CODCs of K_n and the orthogonal labelling.

Theorem 1 *A CODC of K_n by a graph G exists if and only if there exists an orthogonal labelling of G.*

Sampathkumar and Srinivasan [2] called the orthogonal labelling an orthogonal $\{1,2,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\}$-labelling and generalized it to an orthogonal $\{d_1,d_2,\ldots,d_k\}$-labelling, where $\{d_1,d_2,\ldots,d_k\}$ is a sequence of positive integers with $1 \le d_1 \le d_2 \le \ldots \le d_k \le \left\lfloor \frac{n}{2} \right\rfloor$

a) Either n is odd or n is even and $d_k \ne \frac{n}{2}$:

Given a subgraph G of $Circ(n;\{d_1,d_2,\ldots,d_k\})$ with $2k$ edges, a labelling of G, in $\mathbb{Z}_n$, is an orthogonal $\{d_1,d_2,\ldots,d_k\}$-labelling of G if:

(i) for every $l \in \{d_1,d_2,\ldots,d_k\}$, G contains exactly two edges of length l, and

(ii) $\{r(l) : l \in \{d_1,d_2,\ldots,d_k\}\} = \{d_1,d_2,\ldots,d_k\}$.

b) n is even and $d_k = \frac{n}{2}$:

Given a subgraph G of $Circ(n;\{d_1,d_2,\ldots,d_{k-1},\frac{n}{2}\})$ with $2k-1$ edges, a labelling of G, in $\mathbb{Z}_n$, is an orthogonal $\{d_1,d_2,\ldots,d_{k-1},\frac{n}{2}\}$-labelling of G if:

(i) for every $l\in\{d_1,d_2,\ldots,d_{k-1}\}$, G contains exactly two edges of length l and G contains exactly one edge of length $\frac{n}{2}$, and

(ii) $\{r(l):l\in\{d_1,d_2,\ldots,d_{k-1}\}\}=\{d_1,d_2,\ldots,d_{k-1}\}$.

The following theorem of Sampathkumar and Simaringa [2] is a generalization of Theorem 1.

Theorem 2 *A CODC of* $Circ(n;\{d_1,d_2,\ldots,d_k\})$ *by a graph* G *exists if and only if there exists an orthogonal* $\{d_1,d_2,\ldots,d_k\}$ *-labelling of* G.

For results on ODCs of graphs, see [3], a survey by Gronau et al.

In [4], we proved the following. (i) All 3-regular Cayley graphs, except K_4, have ODCs by P_4.

(ii) All 3-regular Cayely graphs on Abelian groups, except K_4, have ODCs by $P_3\cup K_2$. (iii) All 3-regular Cayley graphs on Abelian groups, except K_4 and the 3-prism, have ODCs by $3K_2$.

In [5], Sampathkumar et al. introduced a special kind of orthogonal labelling called orthogonal σ-labelling and they found it for some caterpillars of diameters 4.

In [2], Sampathkumar et al. completely settled the existence problem of CODCs of 4-regular circulant graphs.

Other results of ODCs by different graph classes can be found in [1,2,4,6].

The above results on ODCs of graphs with lower degree motivated me to consider CODCs by certain infinite graph classes which has an orthogonal $\{d_1,d_2,\ldots,d_k\}$-labelling, these graph classes are:

$H_{1,2n}: n>0$, $v\in\mathbb{Z}_{2n}$, graphs consisting of the edges set:

$$E\left(H_{1,2n}\right)=\left\{\{v,v+n+j\},\{v+n+j,v+2j\}:1\le j\le n-1\right\}\cup\{v,v+n\}.$$

$H_{2,2n}: n\ge 2$, $v\in\mathbb{Z}_{2n}$, graphs consisting of $(n-1)C_3$ sharing an edge, whose edges set is

$$E\left(H_{2,2n}\right)=\left\{\left\{v,v+2j\right\},\left\{v+1,v+2j\right\}:1\le j\le n-1\right\}\cup\left\{v,v+1\right\}.$$

$H_{3,n}: n\ge 4$, graphs consisting of $n-1$ vertices $x,\ y,\ z,\ a_i,\ 1\le i\le n-4$ and edges set

$$E\left(H_{3,n}\right)=\left\{\left\{x,y\right\},\left\{y,z\right\},\left\{z,x\right\},\left\{z,a_i\right\}:1\le i\le n-4\right\}.$$

$H_{4,n}: n\ge 8$, graphs consisting of $n-3$ vertices $x,\ y,\ z,\ w,\ u,\ v,\ a_i,\ 1\le i\le n-8$ and edges set

$$E\left(H_{4,n}\right)=\left\{\left\{x,y\right\},\left\{y,v\right\},\left\{x,z\right\},\left\{z,v\right\},\left\{x,u\right\},\left\{u,v\right\},\left\{v,x\right\},\left\{s,a_i\right\}:1\le i\le n-8\right\}.$$

$H_{5,2n}: n\ge 11$, graphs consisting of 11 vertices $v_1,v_2,\ldots,v_{11}$ connected to form two cycles of length 6 where they share a vertex then its edges set is

$$E\left(H_{5,2n}\right)=\begin{Bmatrix}\{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\},\{v_4,v_5\},\{v_5,v_6\},\{v_6,v_1\}\}\cup\\ \{\{v_1,v_7\},\{v_7,v_8\},\{v_8,v_9\},\{v_9,v_{10}\},\{v_{10},v_{11}\},\{v_{11},v_1\}\}\end{Bmatrix}$$

$H_{6,2n}: n\ge 11$, graphs consisting of 12 vertices $v_1,v_2,\ldots,v_{12}$ connected to form two cycles of length 6 and an edge where all of them share a vertex then its edges set is

$$E\left(H_{6,2n}\right)=\begin{Bmatrix}\{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\},\{v_4,v_5\},\{v_5,v_6\},\{v_6,v_1\}\}\cup\\ \{\{v_1,v_7\},\{v_7,v_8\},\{v_8,v_9\},\{v_9,v1_{10}\},\{v_{10},v_{11}\},\{v_{11},v_1\}\}\cup\\ \{\{v_1,v_{12}\}\}\end{Bmatrix}.$$

$H_{7,2n}: n\ge 11$, graphs consisting of 14 vertices $v_1,v_2,\ldots,v_{14}$ connected to form two cycles of length 6 and a star S_3 where they share the center vertex of the star then its edges set is

$$E\left(H_{7,2n}\right)=\begin{Bmatrix}\{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\},\{v_4,v_5\},\{v_5,v_6\},\{v_6,v_1\}\}\cup\\ \{\{v_1,v_7\},\{v_7,v_8\},\{v_8,v_9\},\{v_9,v_{10}\},\{v_{10},v_{11}\},\{v_{11},v_1\}\}\cup\\ \{\{v_1,v_i\}:12\le i\le 14\}\end{Bmatrix}.$$

$H_{8,n}: n\ge 17$, graphs consisting of 11 vertices $v_1,v_2,\ldots,v_{11}$ connected to form two cycles of length 5 where they share a vertex and each of which connected to an edge then its edges set is

$$E\left(H_{8,n}\right)=\begin{Bmatrix}\{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\},\{v_4,v_5\},\{v_5,v_1\}\}\cup\\ \{\{v_1,v_6\},\{v_6,v_7\},\{v_7,v_8\},\{v_8,v_9\},\{v_9,v_1\}\}\cup\\ \{\{v_9,v_{10}\},\{v_5,v_{11}\}\}\end{Bmatrix}.$$

Finally, for any positive integers m and n, let G be a graph that has an orthogonal $\{d_1,d_2,\ldots,d_k\}$ -labelling, then the edges of length $d_i\in A=\{d_1,d_2,\ldots,d_k\}$ are $\{u_{d_i},u_{d_i}+d_i\}$ and $\{u_{-d_i},u_{-d_i}-d_i\}$ where $i\in\{1,2,\ldots,k\}$ and u_{d_i}, $u_{-d_i}\in\mathbb{Z}_n$. Also, let H be a graph that has an orthogonal $\{t_1,t_2,\ldots,t_s\}$ -labelling, then the edges of length $t_j\in B=\{t_1,t_2,\ldots,t_s\}$ are $\{v_{t_j},v_{t_j}+t_j\}$ and $\{v_{-t_j},v_{-t_j}-t_j\}$ where $j\in\{1,2,\ldots,s\}$ and v_{t_j}, $v_{-t_j}\in\mathbb{Z}_m$. Then let us define a new graph $P(G,H)$ to be the graph with edges set $\left\{\left\{\left(u_{d_i},v_{t_j}\right),\left(u_{d_i}+d_i,v_{t_j}+t_j\right)\right\}:d_i\in A \text{ and } t_j\in B\right\}$. For this definition, Theorem 11 can be deduced.

2 CODCs by Certain Infinite Graph Classes

Theorem 3 *For any positive integer* n *, there exists a CODC of* $(2n-1)$ *-regular* $Circ(2n,\{1,2,\ldots,n\})$ *by* $H_{1,2n}$.

Proof. In $H_{1,2n}$, the edge of length n is $\{v,v+n\}$; the other lengths are the elements of $\{|n+j|:1<j\le n-1\}=\{1,2,\ldots,n-1\}$ and $\{|n-j|:1\le j\le n-1\}=\{1,2,\ldots,n-1\}$, then (i) for every $l\in\{1,2,\ldots,n-1\}$, $H_{1,2n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\{r(l):l\in\{1,2,\ldots,n-1\}\}=\{1,2,\ldots,n-1\}$.

From (i) and (ii), $H_{1,2n}$ has an orthogonal $\{1,2,\ldots,n\}$ -labelling.■

Theorem 4 *For any positive integer* n *, there exists a CODC of* $(2n-1)$ *-regular* $Circ(2n,\{1,2,\ldots,n\})$ *by* $H_{2,2n}$.

Proof. In $H_{2,2n}$,

Case 1. n is even.
The edge of length

n is $\{v, v+n\}$ and the edges of lengths $\left\{|2j| : 1 \le j \le \frac{n}{2} - 1\right\}$ are $\left\{\{v, v+2j\} : 1 \le j \le \frac{n}{2} - 1\right\}$ and $\left\{\{v, v+2j\} : \frac{n}{2} + 1 \le j \le n-1\right\}$; ones of lengths $\left\{|2j-1| : 1 \le j \le \frac{n}{2} - 1\right\}$ are

$\left\{\{v+1, v+2j\} : 1 \le j \le \frac{n}{2} - 1\right\}$ and $\left\{\{v+1, v+2j\} : \frac{n}{2} + 1 \le j \le n-1\right\}$, then (i) for every $l \in \{1, 2, \dots, n-1\}$, $H_{2,2n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\{r(l) : l \in \{1, 2, \dots, n-1\}\} = \{1, 2, \dots, n-1\}$.

Case 2. n is odd.

The edge of length n is $\{v+1, n+1\}$ and the edges of lengths $\left\{|2j| : 1 \le j \le \frac{n-1}{2}\right\}$ are $\left\{\{v, v+2j\} : 1 \le j \le \frac{n-1}{2}\right\}$ and $\left\{\{v, v+2j\} : \frac{n+1}{2} \le j \le n-1\right\}$; ones of lengths $\left\{|2j-1| : 1 \le j \le \frac{n-1}{2}\right\}$ are $\left\{\{v+1, v+2j\} : 1 \le j \le \frac{n-1}{2}\right\}$ and $\left\{\{v+1, v+2j\} : \frac{n+1}{2} \le j \le n-1\right\}$, then (i) for every $l \in \{1, 2, \dots, n-1\}$, $H_{2,2n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\{r(l) : l \in \{1, 2, \dots, n-1\}\} = \{1, 2, \dots, n-1\}$. From (i) and (ii), $H_{2,2n}$ has an orthogonal $\{1, 2, \dots, n\}$-labelling.■

Theorem 5 *For any positive integer* $n \ge 4$ *, there exists a CODC of* $(n-1)$ *-regular* $Circ(n, \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\})$ *by* $H_{3,n}$.

Proof. Consider the following labelling ψ of $H_{3,n} : \psi(x) = 0;\ \ \psi(y) = 1;\ \ \psi(z) = 2;$ $\psi(a_i) = i + 3$ if $1 \le i \le n-4$.

Case 1. n is even.

The edge of length $\frac{n}{2}$ is $\left\{2,\frac{n}{2}+2\right\}$; ones of length 1 are $\{0,1\}$ and $\{1,2\}$; ones of length l where $2\le l\le\frac{n}{2}-1$ are $\left\{\{2,j\}:4\le j\le\frac{n}{2}+1\right\}$ and $\left\{\{2,j\}:\frac{n}{2}+3\le j\le n-1\right\}$, then (i) for every $l\in\left\{1,2,\ldots,\frac{n}{2}-1\right\}$, $H_{3,n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\left\{r(l):l\in\left\{1,2,\ldots,\frac{n}{2}-1\right\}\right\}=\left\{1,2,\ldots,\frac{n}{2}-1\right\}$.

Case 2. n is odd.

The edges of length 1 are $\{0,1\}$ and $\{1,2\}$; ones of length l where $2\le l\le\frac{n-1}{2}$ are $\left\{\{2,j\}:4\le j\le\frac{n+3}{2}\right\}$ and $\left\{\{2,j\}:\frac{n+5}{2}\le j\le n-1\right\}$, then (i) for every $l\in\left\{1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$, $H_{3,n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\left\{r(l):l\in\{1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\}\right\}=\left\{1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$. ■

Theorem 6 *For any positive integer* $n\ge 8$ *, there exists a CODC of* $(n-1)$ *-regular* $Circ(n,\{1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\})$ *by* $H_{4,n}$.

Proof. Consider the following labelling ψ of $H_{4,n}$: $\psi(x)=0$; $\psi(y)=1$; $\psi(z)=2$; $\psi(u)=n-1$; $\psi(v)=4$; $\psi(a_i)=i+6$ if $1\le i\le n-8$.

Case 1. n is even.

The edge of length $\frac{n}{2}$ is $\left\{4,\frac{n}{2}+4\right\}$; ones of length 1 are $\{0,1\}$ and $\{0,n-1\}$; ones of length 2 are $\{0,2\}$ and $\{2,4\}$ ones of length l where $3\le l\le\frac{n}{2}-1$ are $\left\{\{4,j\}:7\le j\le\frac{n}{2}+3\right\}$ and $\left\{\{4,j\}:\frac{n}{2}+5\le j\le n-2\right\}$, then (i) for every

$l \in \left\{1, 2, \ldots, \frac{n}{2} - 1\right\}$, $H_{4,n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\left\{r(l) : l \in \left\{1, 2, \ldots, \frac{n}{2} - 1\right\}\right\} = \left\{1, 2, \ldots, \frac{n}{2} - 1\right\}$.

Case 2. n is odd.

The edges of length 1 are $\{0,1\}$ and $\{0, n-1\}$; ones of length 2 are $\{0,2\}$ and $\{2,4\}$ ones of length l where $3 \le l \le \frac{n-1}{2}$ are $\left\{\{4, j\} : 7 \le j \le \frac{n+7}{2}\right\}$ and $\left\{\{4, j\} : \frac{n+9}{2} \le j \le n-2\right\}$, then (i) for every $l \in \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, $H_{4,n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\left\{r(l) : l \in \left\{1, 2, \ldots, \frac{n}{2} - 1\right\}\right\} = \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$.■

Theorem 7 *For any positive integer* $n \ge 11$ *, there exists a CODC of the 12-regular* $Circ\left(2n, \{1, 2, 3, 4, n-6, n-4\}\right)$ *by* $H_{5,2n}$.

Proof. Consider the following labelling ψ of $H_{5,2n}$: $\psi(v_1) = 0$; $\psi(v_2) = 1$; $\psi(v_3) = 4$; $\psi(v_4) = 7$; $\psi(v_5) = n+3$; $\psi(v_6) = 2n-1$; $\psi(v_7) = 2$; $\psi(v_8) = 6$; $\psi(v_9) = 10$; $\psi(v_{10}) = n+4$; $\psi(v_{11}) = 2n-2$.

Then the edges of length 1 are $\{0,1\}$ and $\{0, 2n-1\}$; ones of length 2 are $\{0,2\}$ and $\{0, 2n-2\}$; ones of length 3 are $\{1,4\}$ and $\{4,7\}$; ones of length 4 are $\{2,6\}$ and $\{6,10\}$; ones of length $n-6$ are $\{10, n+4\}$ and $\{n+4, 2n-2\}$; ones of length $n-4$ are $\{7, n+3\}$ and $\{n+3, 2n-1\}$, then (i) for every $l \in \{1, 2, 3, 4, n-6, n-4\}$, $H_{5,2n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\{r(l) : l \in \{1, 2, 3, 4, n-6, n-4\}\} = \{1, 2, 3, 4, n-6, n-4\}$.■

Theorem 8 *For any positive integer* $n \ge 11$ *, there exists a CODC of the 13-regular* $Circ\left(2n, \{1, 2, 3, 4, n-6, n-4, n\}\right)$ *by* $H_{6,2n}$.

Proof. Consider the following labelling ψ of $H_{6,2n}$: $\psi(v_1)=0$; $\psi(v_2)=1$; $\psi(v_3)=4$; $\psi(v_4)=7$; $\psi(v_5)=n+3$; $\psi(v_6)=2n-1$; $\psi(v_7)=2$; $\psi(v_8)=6$; $\psi(v_9)=10$; $\psi(v_{10})=n+4$; $\psi(v_{11})=2n-2$; $\psi(v_{12})=n$.

Then the edge of length n is $\{0,n\}$; ones of length 1 are $\{0,1\}$ and $\{0,2n-1\}$; ones of length 2 are $\{0,2\}$ and $\{0,2n-2\}$; ones of length 3 are $\{1,4\}$ and $\{4,7\}$; ones of length 4 are $\{2,6\}$ and $\{6,10\}$; ones of length $n-6$ are $\{10,n+4\}$ and $\{n+4,2n-2\}$; ones of length $n-4$ are $\{7,n+3\}$ and $\{n+3,2n-1\}$, then (i) for every $l\in\{1,2,3,4,n-6,n-4\}$, $H_{6,2n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\{r(l):l\in\{1,2,3,4,n-6,n-4\}\}=\{1,2,3,4,n-6,n-4\}$.■

Theorem 9 *For any positive integer* $n\geq 11$ *, there exists a CODC of the 15-regular* $Circ(2n,\{1,2,3,4,5,n-6,n-4,n\})$ *by* $H_{7,2n}$.

Proof. Consider the following labeling ψ of $H_{7,2n}$:
$\psi(v_1)=0$; $\psi(v_2)=1$; $\psi(v_3)=4$; $\psi(v_4)=7$; $\psi(v_5)=n+3$; $\psi(v_6)=2n-1$; $\psi(v_7)=2$; $\psi(v_8)=6$; $\psi(v_9)=10$; $\psi(v_{10})=n+4$; $\psi(v_{11})=2n-2$; $\psi(v_{12})=5$; $\psi(v_{13})=n$; $\psi(v_{14})=2n-5$.

Then the edge of length n is $\{0,n\}$; ones of length 1 are $\{0,1\}$ and $\{0,2n-1\}$; ones of length 2 are $\{0,2\}$ and $\{0,2n-2\}$ ones of length 3 are $\{1,4\}$ and $\{4,7\}$; ones of length 4 are $\{2,6\}$ and $\{6,10\}$; ones of length 5 are $\{0,5\}$ and $\{0,2n-5\}$; ones of length $n-6$ are $\{10,n+4\}$ and $\{n+4,2n-2\}$; ones of length $n-4$ are $\{7,n+3\}$ and $\{n+3,2n-1\}$, then (i) for every $l\in\{1,2,3,4,5,n-6,n-4\}$, $H_{7,2n}$ contains exactly two edges of length l, and (ii) since every two edges of the same length are adjacent then $\{r(l):l\in\{1,2,3,4,5,n-6,n-4\}\}=\{1,2,3,4,5,n-6,n-4\}$.■

Theorem 10 *For any positive integer* $n\geq 17$ *, there exists a CODC of the 12-regular* $Circ(n,\{1,2,3,4,n-12,n-8\})$ *by* $H_{8,n}$.

Proof. Consider the following labelling ψ of $H_{8,n}$: $\psi(v_1)=0$; $\psi(v_2)=1$; $\psi(v_3)=4$; $\psi(v_4)=7$; $\psi(v_5)=n-1$; $\psi(v_6)=2$; $\psi(v_7)=6$; $\psi(v_8)=10$; $\psi(v_9)=n-2$; $\psi(v_{10})=n-14$; $\psi(v_{11})=n-9$.

Then the edges of length 1 are $\{0,1\}$ and $\{0,n-1\}$; ones of length 2 are $\{0,2\}$ and $\{0,n-2\}$; ones of length 3 are $\{1,4\}$ and $\{4,7\}$; ones of length 4 are $\{2,6\}$ and $\{6,10\}$; ones of length $n-12$ are $\{10,n-2\}$ and $\{n-14,n-2\}$; ones of length $n-8$ are $\{7,n-1\}$ and $\{n-1,n-9\}$, then (i) for every $l\in\{1,2,3,4,n-12,n-8\}$, $H_{8,n}$ contains exactly two edges of length l , and (ii) since every two edges of the same length are adjacent then $\{r(l):l\in\{1,2,3,4,n-12,n-8\}\}=\{1,2,3,4,n-12,n-8\}$.■

Theorem 11 *For any positive integers* m *and* n *there exists a CODC of* $4|A||B|$ *-regular* $Circ(mn,\ A\times B)$ *by* $P(G,H)$ *with respect to* $\mathbb{Z}_n\times\mathbb{Z}_{m}$*.*

Proof. Since G and H have Orthogonal A -labellings and Orthogonal B -labellings respectively then the two edges of length (d_i,t_j) in $P(G,H)$ are $\{(u_{d_i},v_{t_j}),(u_{d_i}+d_i,v_{t_j}+t_j)\}$ and $\{(u_{-d_i},v_{-t_j}),(u_{-d_i}-d_i,v_{-t_j}-t_j)\}$ and the set of all rotation distances will be $A\times B$. Then $P(G,H)$ has orthogonal $A\times B$ -labellings with respect to $\mathbb{Z}_n\times\mathbb{Z}_{m}$.

3 Conclusion

In this paper, the existences of the CODCs using certain infinite classes of graphs are completely settled (see Theorem 3 to Theorem 11).

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Competing Interests

Author has declared that no competing interests exist.

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