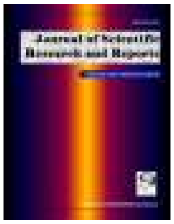

Journal of Scientific Research & Reports
4(4): 328-347, 2015; Article no.JSRR.2015.036
ISSN: 2320–0227

SCIENCEDOMAIN *international*
www.sciencedomain.org

TL-Moments and LQ-Moments of the Exponentiated Pareto Distribution

Samir K. Ashour[1*], Ahmed A. El-sheik[1] and Noura A. T. Abu El-Magd[2]**

[1]Institute of Statistical Studies and Research, Cairo University, Giza, Egypt.
[2]Faculty of Business and Economics, Misr University for Science and Technology, Giza, Egypt.

Authors' contributions

This work was carried out in collaboration between all authors. Author SKA designed the study, wrote the protocol, and wrote the first draft of the manuscript. Author AAE managed the literature searches, analyses of the study performed the spectroscopy analysis and author NATAE managed the experimental process and identified the species of plant. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JSRR/2015/12047
Editor(s):
(1) Masum A Patwary, Geography and Environmental Science, Begum Rokeya University Rangpur 5400, Bangladesh.
Reviewers:
(1) Anonymous, Sapienza Università di Roma, Roma, Italy.
(2) Gerhard-Wilhelm Weber, Institute of Applied Mathematics, Middle East Technical University, Germany.
Complete Peer review History: http://www.sciencedomain.org/review-history.php?iid=743&id=22&aid=6604

Original Research Article

Received 16th June 2014
Accepted 23rd August 2014
Published 23rd October 2014

ABSTRACT

The main concern of this paper is to derive the TL-moments (Trimmed Linear moments) of the exponentiated Pareto distribution (EPD) and use the TL-moments to estimate the unknown parameters of the EPD. Many special cases may be obtained such as L-moments (Linear moments), LH-moments (Linear Higher moments) and LL-moments (Linear Lower moments). Also, the LQ-moments (Linear Quantile moments) with the three cases (trimean, median and Gastwirth) will be obtained and used to estimate the unknown parameters of the EPD. The estimation of the EPD parameters is studied in numerical simulations where the method for obtaining TL-moment estimators is compared with other estimation methods (L-moment estimators, LQ-moment [trimean, median and Gastwirth] estimators, maximum likelihood estimators and the method of moment estimators). According to these comparisons, it is suggested that the method of L-moments is preferable for small sample size.

---

*Corresponding authors: E-mail: *Ashoursamir@Hotmail.com, Shon_stat@Hotmail.com*

Keywords: Exponentiated pareto distribution; traditional moments; TL-moments; L-moments; LH-moments; LL-moments; LQ-moments; order statistics; simulations; maximum likelihood.

1. INTRODUCTION

The exponentiated Pareto distribution (EPD) is an important extension of the Pareto family as a lifetime model and can be used quite effectively in analyzing many lifetime data. The EPD is a generalized version of the Pareto distribution. EPD has the following cumulative distribution function:

$$F(x) = \left(1-(1+x)^{-\eta}\right)^{\alpha}, \ x>0, \ \alpha>0, \ \eta>0, \tag{1-1}$$

where $\alpha > 0$ and $\eta > 0$ are two shape parameters. The corresponding density function will be:

$$f(x) = \alpha\eta\left(1-(1+x)^{-\eta}\right)^{\alpha-1}(1+x)^{-(\eta+1)}, \ x>0, \ \alpha>0, \ \eta>0 \tag{1-2}$$

If $\alpha = 1$, the above distribution reduces to the standard Pareto distribution of the second kind. Gupta et al. [1] obtained the quantile function for the EPD as follows:

$$Q(u) = \left(1-u^{1/\alpha}\right)^{-1/\eta} - 1, \quad 0<u<1. \tag{1-3}$$

Shawky and Abu-Zinadah (King Abdulaziz University, Jeddah, Saudi Arabia, Unpublished results, 2006) studied EPD and studied some of its mathematical properties and obtained the r[th] classical moments (Traditional moments) as follows:

$$\mu_r = \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}A_j(\alpha) + (-1)^r, \ \eta>r, \ r=0, 1, 2, 3, ..., \tag{1-4}$$

where

$$A_j(\alpha) = \alpha \mathrm{B}(\alpha, \ 1-j/\eta), \ \eta > j$$

where $B(\alpha, 1-j/\eta)$ is a beta function with $\eta > j$. The above closed form of μ_r allows them to derive the following statistical measures for the EPD, the mean and the variance of the EPD will be:

$$\mu = A_1(\alpha)-1, \ \eta>1, \ and \ \sigma^2 = A_2(\alpha) - A_1^2(\alpha), \ \eta>2. \tag{1-5}$$

They derived the skewness, kurtosis and the coefficient of variation for the EPD. The coefficient of variation (CV) is:

$$CV = \frac{\sqrt{A_2(\alpha)-A_1^2(\alpha)}}{A_1(\alpha)-1}, \ \eta>2. \tag{1-6}$$

The measure of skewness and kurtosis, respectively are:

$$\alpha_3 = \frac{A_3(\alpha) - A_1(\alpha)\left(2A_1^2(\alpha) - 3A_2(\alpha)\right)}{\left(A_2(\alpha)-A_1^2(\alpha)\right)^{3/2}}, \ \eta>3, \tag{1-7}$$

and

$$\alpha_4 = \frac{A_4(\alpha) + A_1(\alpha)\left(-4A_3(\alpha) + 6A_2(\alpha)A_1(\alpha) - 3A_1^3(\alpha)\right)}{\left(A_2(\alpha)-A_1^2(\alpha)\right)^2}, \ \eta>4. \tag{1-8}$$

Shawky and Abu-Zinadah [2] estimated the unknown parameters of the EPD by the maximum likelihood method. Also, they considered five other estimation procedures (method of moment estimators, estimators based on percentiles, least squares estimators, weighted least squares estimators, L-moments estimators) and compared their performances through numerical simulations with respect to their biases and root mean squared errors (RMSEs). Compering the performance of all the estimators, they concluded that the maximum likelihood estimator performs best in most cases considered. Interestingly, while estimating η, the biases and RMSEs of the L-moments estimators are lower than the other estimators most of the times. They recommended to use the maximum likelihood estimator for estimating α and η when both are unknown.

Hosking [3] introduced the concept of the linear moments (L-moments) and concluded that L-moments of a probability distribution to be meaningful, we require only that the distribution has a finite mean; for standard errors of L-moments to be finite, we require only that the distribution has a finite variance and L-moments are being linear functions of the data, are less sensitive than are classical moments to sampling variability or measurement errors in the extreme data values.

Elamir and Seheult [4] introduced the trimmed linear moments (TL-moments) and concluded that TL-moments are more resistant to outliers, TL-Moments assign zero weight to the extreme observations and they are easy to compute and a population TL-Moments may be well defined where the corresponding population L-Moments (or central moment) does not exist. Mudholkar and Hutson [5] introduced the concept of the linear quantile moments (LQ-moments) and concluded that LQ-moments are often easier to evaluate and estimate than L-moments, LQ-moments always exist and unique and their asymptotic distributions are easier to obtain.

Abu El-Magd [6] introduced the theoretical comparison between the TL-moments, L-moments and LQ-moments with explain the advantages of each moments with numerical comparison for the exponentiated generalized extreme value (EGEV) distribution. He introduced the TL-moment and LQ-moment estimators of the EGEV distribution with a numerical simulation. He compared the TL-moment estimators with other estimation methods (L-moment estimators, LQ-moment estimators and the method of moment estimators) mainly with respect to their biases and root mean squared errors (RMSEs). He recommended using the TL-moment estimators for estimating the parameters of EGEV distribution for large sample size (n =50, 100) and recommended using the LQ-moment estimators for small sample size (n = 15, 25).

Zaher et al. [7] obtained the fuzzy least-squares estimator for the two-parameter Pareto distribution. Also, they obtained the TL-moments, L-moments and LQ-moments formulas for the two-parameter Pareto distribution. Numerical comparisons between fuzzy least-squares estimators and the different estimators for the two-parameter Pareto distribution are implemented. They suggested that the fuzzy least-squares estimator is preferable all times.

In this paper, two different types of moments namely TL-moments and LQ-moments (with the three different cases median, trimean and Gastwirth) of the EPD will be derived and used to estimate the unknown parameters of EPD. Different special cases {TL-moments with the first trimmed, L-moments, Linear higher moments (LH-moments) and Linear lower moments (LL-moments)} for the EPD will be obtained. Algorithms are suggested for parameters estimation and a numerical illustration for the new results will be given. Numerical comparisons between seven estimators {maximum likelihood estimators, method of moment estimators, TL-moment estimators, L-moment estimators and LQ-moment estimators (median, trimean and Gastwirth)} will be carried out.

In section 2, the maximum likelihood estimators for the unknown parameters of the EPD will be reviewed and in section 3, the method of moment estimators for the unknown parameters of the EPD will be reviewed. In section 4, the TL-moments formulas will be obtained for the EPD with many special cases of the EPD. Also, the TL-moment estimators and the L-moment estimators for the unknown parameters of the EPD will be derived. In section 5, the LQ-moments formulas with different cases (median, trimean and Gastwirth) will be obtained for the EPD. Also, the LQ-moment estimators with different cases for the unknown parameters of the EPD will be derived. In section 6, a simulation study is conducted. Finally, the results and conclusions will be introduced for the EPD in section 7.

2. MAXIMUM LIKELIHOOD ESTIMATORS FOR (MLEs) THE EPD

Maximum Likelihood estimators (MLEs) are those values of the parameters that maximize the log likelihood function. Shawky and Abu-Zinadah [2] introduced the MLEs of the EPD with two different cases. First, they considered the estimation of α and η when both are unknown. If $x_1, x_2, \ldots, x_n$ is a random sample of size n from the EPD, then the log-likelihood function for the EPD is given by:

$$\ln L = n\ln\alpha + n\ln\eta + (\alpha-1)\sum_{i=1}^{n}\ln\left(1-(1+x_i)^{-\eta}\right) - (\eta+1)\sum_{i=1}^{n}\ln(1+x_i). \qquad (2\text{-}1)$$

To maximize function (2-1), they obtained the partial derivatives of the function with respect to each parameter, and then set the resulting two equations equal to 0 and solve the equations for the two parameters. The partial derivatives are:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left(1 - (1 + x_i)^{-\eta}\right), \tag{2-2}$$

and

$$\frac{\partial \ln L}{\partial \eta} = \frac{n}{\eta} + (\alpha - 1)\sum_{i=1}^{n} \frac{(1 + x_i)^{-\eta}}{\left(1 - (1 + x_i)^{-\eta}\right)} \ln(1 + x_i) - \sum_{i=1}^{n} \ln(1 + x_i). \tag{2-3}$$

The MLEs are the values of the parameters that satisfy:

$$\frac{\partial \ln L}{\partial \alpha} = 0, \ \ and \ \ \frac{\partial \ln L}{\partial \eta} = 0 \tag{2-4}$$

Then, they obtained:

$$\alpha^* = -n\left(\sum_{i=1}^{n} \ln\left(1 - (1 + x_i)^{-\eta^*}\right)\right)^{-1} \tag{2-5}$$

Putting α^* in (2-1), they obtained:

$$g(\eta) = n \ln n - n \ln \sum_{i=1}^{n} \left(-\ln\left(1 - (1 + x_i)^{-\eta}\right)\right) + n \ln \eta - n - \sum_{i=1}^{n} \ln\left(1 - (1 + x_i)^{-\eta}\right) - (\eta + 1)\sum_{i=1}^{n} \ln(1 + x_i). \tag{2-6}$$

Therefore, the estimator η^* can be obtained by maximizing $g(\eta)$ with respect to η. They observed that $g(\eta)$ is a unimodal function, and η^* which maximizes $g(\eta)$ can be obtained from the fixed point solution of $h(\eta) = \eta$ where

$$h(\eta) = \left(\frac{\sum_{i=1}^{n} (1 + x_i)^{-\eta} \ln(1 + x_i) \Big/ \left(1 - (1 + x_i)^{-\eta}\right)}{\sum_{i=1}^{n} \ln\left(1 - (1 + x_i)^{-\eta}\right)} + \frac{1}{n}\sum_{i=1}^{n} \frac{\ln(1 + x_i)}{\left(1 - (1 + x_i)^{-\eta}\right)}\right)^{-1}. \tag{2-7}$$

They obtained that a very simple iterative procedure can be used to find a solution of $h(\eta) = \eta$, and it worked very well. Once they obtained η^*, the estimator α^* can be obtained from (2-5).

3. METHOD OF MOMENT ESTIMATORS (MMEs) FOR THE EPD

Shawky and Abu-Zinadah [2] introduced the method of moment estimators (MMEs) of the unknown parameters of the EPD. First, they considered the estimation of α and η when both are unknown. For the EPD, they have two parameters, so they require the first two sample moments: sample mean and variance. These sample moments are equated to their population analogues, and the resulting equations are:

$$\bar{x} = \alpha^{**} \mathrm{B}\left(\alpha^{**}, 1 - \frac{1}{\eta^{**}}\right) - 1, \ \ \eta^{**} > 1, \tag{3-1}$$

and

$$s^2 = \alpha^{**} \mathrm{B}\left(\alpha^{**}, 1 - \frac{2}{\eta^{**}}\right) - \left(\alpha^{**} \mathrm{B}\left(\alpha^{**}, 1 - \frac{1}{\eta^{**}}\right)\right)^2, \ \ \eta^{**} > 2 \tag{3-2}$$

Where $\bar{x}$ and s^2 are the sample mean and the sample variance, respectively. Then, the MMEs of α and η, say α^{**} and η^{**}, respectively, can be obtained by solving the two equations (3-1) and (3-2). Clearly, it was not possible to obtain the exact variances of α^{**} and η^{**}.

When the shape parameter η is known, they obtained the MME of α can be obtained by solving the non-linear equation (3-1) with respect to α. In the same way, if the shape parameter α is known, then the MME of η can be obtained by solving the nonlinear equation (3-2) with respect to η

4. TL-MOMENTS ESTIMATORS (TLMEs) FOR THE EPD

In this section, the TL-moments with generalized trimmed for the EPD will be obtained. Many special cases from the TL-moments with generalized trimmed will be obtained for the EPD such as the TL-moments with the first trimmed, L-moments, LL-moments and LH-moments. The TL-moments estimators with the first trimmed (TLMEs) and L-moments estimators (LMEs) will be derived for the unknown parameters of the EPD.

4.1 TL-Moments

Let, $X_1, X_2, ..., X_n$ be a conceptual random sample (used to define a population quantity) of size n from a continuous distribution and let, $X_{(1:n)} \le X_{(2:n)} \le ... \le X_{(n:n)}$ denote the corresponding order statistics. Elamir and Seheult [4] defined the r^th population TL-moment $\lambda_r^{(s,t)}$ as follows:

$$\lambda_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left(X_{(r+s-k:r+s+t)}\right), \quad r = 1,2,... \tag{4-1}$$

where s, t = 0, 1, 2,…., they considered the symmetric case $s = t$. Hosking [8] obtained TL-moments with generalized trimmed for s and t (symmetric case $s = t$ and asymmetric case $s \ne t$) and obtained the TL-moments coefficient of variation (TL-CV), the TL-skewness and the TL-kurtosis as follows:

$$\tau^{(s,t)} = \lambda_2^{(s,t)} \big/ \lambda_1^{(s,t)}, \ \tau_3^{(s,t)} = \lambda_3^{(s,t)} \big/ \lambda_2^{(s,t)}, \ and \ \tau_4^{(s,t)} = \lambda_4^{(s,t)} \big/ \lambda_2^{(s,t)} \tag{4-2}$$

Maillet and M´edecin [9] introduced the relation between the r^th TL-moments and the first TL-moments with generalized trimmed for s and t (for the two cases: symmetric case $(s = t)$ and asymmetric case $(s \ne t)$). They obtained the r^th TL-moments by using the quantile function Q(u) and as follows:

$$\lambda_r^{(s,t)} = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \frac{(r+s+t)!}{(r+s-j-1)!(t+j)!}\int_0^1 Q(u)u^{r+s-j-1}(1-u)^{t+j}\,du, \quad r = 1,\ 2,\ ..., \tag{4-3}$$

Using the r^th TL-moments, they introduced an important relation between the first TL-moment $\lambda_1^{(s,t)}$ and the r^th TL-moments $\lambda_r^{(s,t)}$ as follows:

$$\lambda_r^{(s,t)} = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \lambda_1^{(r+s-j-1,t+j)}, \quad r = 1,\ 2,\ ,3,\ ... \tag{4-4}$$

where s, t = 0, 1, 2,….. . Using the quantile function of the EPD (1-3), the first TL-moments with generalized trimmed $\lambda_1^{(s,t)}$ will be:

$$\lambda_1^{(s,t)} = \frac{(s+t+1)!}{(s)!(t)!}\int_0^1 Q(u)u^s(1-u)^t\,du,$$
$$= C\int_0^1 \left(\left(1-u^{1/\alpha}\right)^{-1/\eta}-1\right)u^s(1-u)^t\,du$$
$$= C\int_0^1 \left(1-u^{1/\alpha}\right)^{-1/\eta}u^s(1-u)^t\,du - C\int_0^1 u^s(1-u)^t\,du,$$
$$= C\int_0^1 \left(1-u^{1/\alpha}\right)^{-1/\eta}u^s(1-u)^t\,du - C\mathrm{B}(s+1,t+1),$$
$$= C\int_0^1 \left(1-u^{1/\alpha}\right)^{-1/\eta}u^s(1-u)^t\,du - 1. \tag{4-5}$$

where, $C=\frac{(s+t+1)!}{(s)!(t)!}$, and $(1-u)^t=\sum_{k=0}^{t}(-1)^k\binom{t}{k}u^k$, then, (4-5) will be:

$$\lambda_1^{(s,t)} = C\sum_{k=0}^{t}(-1)^k\binom{t}{k}\int_0^1\left(1-u^{1/\alpha}\right)^{-1/\eta}u^{s+k}\,du-1. \tag{4-6}$$

Let, $y=u^{1/\alpha}$, then, (4-6) will be:

$$\lambda_1^{(s,t)} = C\sum_{k=0}^{t}(-1)^k\binom{t}{k}\int_0^1 \alpha y^{\alpha(s+k)+\alpha-1}\left(1-y\right)^{-1/\eta}dy-1$$
$$= C\alpha\sum_{k=0}^{t}(-1)^k\binom{t}{k}\int_0^1 y^{\alpha(s+k+1)-1}\left(1-y\right)^{-1/\eta}dy-1,$$
$$= C\alpha\sum_{k=0}^{t}(-1)^k\binom{t}{k}\int_0^1 y^{\alpha(s+k+1)-1}\left(1-y\right)^{1-1/\eta-1}dy-1,$$
$$= C\alpha\sum_{k=0}^{t}(-1)^k\binom{t}{k}\mathrm{B}\left(\alpha(s+k+1),1-\frac{1}{\eta}\right)-1, \tag{4-7}$$

Hence, the first TL-moments with generalized trimmed for s and t will be:

$$\lambda_1^{(s,t)} = \frac{(s+t+1)!}{(s)!(t)!}\alpha\sum_{k=0}^{t}(-1)^k\binom{t}{k}\mathrm{B}\left(\alpha(s+k+1),1-\frac{1}{\eta}\right)-1. \tag{4-8}$$

Now, we will introduce the second, third and fourth TL-moments with generalized trimmed for any positive integer s and t for the EPD by using the relation (4-4) between $\lambda_r^{(s,t)}$ and $\lambda_1^{(r+s-j-1,t+j)}$, we can obtain the second TL-moments with generalized trimmed for s and t as follows:

$$\lambda_2^{(s,t)} = \frac{1}{2}\left[\lambda_1^{(s+1,t)}-\lambda_1^{(s,t+1)}\right], \tag{4-9}$$

and by using the formula (4-8) for $\lambda_1^{(s,t)}$, we will obtain:

$$\lambda_2^{(s,t)} = \frac{1}{2}\alpha\left[\frac{(s+t+2)!}{(s+1)!(t)!}\sum_{k=0}^{t}(-1)^k\binom{t}{k}\mathrm{B}\left(\alpha(s+k+2),1-\frac{1}{\eta}\right)\right.$$
$$\left.-\frac{(s+t+2)!}{(s)!(t+1)!}\sum_{k=0}^{t+1}(-1)^k\binom{t+1}{k}\mathrm{B}\left(\alpha(s+k+1),1-\frac{1}{\eta}\right)\right],$$
$$=\frac{1}{2}\alpha\frac{(s+t+2)!}{(s+1)!}\left[\sum_{k=0}^{t}\frac{(-1)^k}{(k)!(t-k)!}\mathrm{B}\left(\alpha(s+k+2),1-\frac{1}{\eta}\right)\right.$$
$$\left.-\frac{(s+1)}{(t+1)!}\left(\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)+\sum_{k=0}^{t}(-1)^{k+1}\binom{t+1}{k+1}\mathrm{B}\left(\alpha(s+k+2),1-\frac{1}{\eta}\right)\right)\right],$$
$$=\frac{1}{2}\alpha\frac{(s+t+2)!}{(s+1)!}\left[-\frac{(s+1)}{(t+1)!}\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)+\sum_{k=0}^{t}\frac{(-1)^k}{(k)!(t-k)!}\mathrm{B}\left(\alpha(s+k+2),1-\frac{1}{\eta}\right)\right.$$
$$\left.+\sum_{k=0}^{t}\frac{(-1)^k(s+1)}{(k+1)!(t-k)!}\mathrm{B}\left(\alpha(s+k+2),1-\frac{1}{\eta}\right)\right], \qquad (4\text{-}10)$$

Hence, the second TL-moments with generalized trimmed for s and t will be:

$$\lambda_2^{(s,t)} = \frac{1}{2}\alpha\frac{(s+t+2)!}{(s+1)!}\left[-\frac{(s+1)}{(t+1)!}\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)\right.$$
$$\left.+\sum_{k=0}^{t}\frac{(-1)^k(s+k+2)}{(k+1)!(t-k)!}\mathrm{B}\left(\alpha(s+k+2),1-\frac{1}{\eta}\right)\right]. \qquad (4\text{-}11)$$

Similarly, using the same relation (4-4), we can also obtain the third TL-moments with generalized trimmed for any positive integer s and t for the EPD as follows:

$$\lambda_3^{(s,t)} = \frac{1}{3}\left[\lambda_1^{(s+2,t)} - 2\lambda_1^{(s+1,t+1)} + \lambda_1^{(s,t+2)}\right], \qquad (4\text{-}12)$$

and using the formula (4-8) of $\lambda_1^{(s,t)}$, then we have:

$$\lambda_3^{(s,t)} = \frac{1}{3}\alpha\left[\frac{(s+t+3)!}{(s+2)!(t)!}\sum_{k=0}^{t}(-1)^k\binom{t}{k}\mathrm{B}\left(\alpha(s+k+3),1-\frac{1}{\eta}\right)\right.$$
$$-2\frac{(s+t+3)!}{(s+1)!(t+1)!}\sum_{k=0}^{t+1}(-1)^k\binom{t+1}{k}\mathrm{B}\left(\alpha(s+k+2),1-\frac{1}{\eta}\right)$$
$$\left.+\frac{(s+t+3)!}{(s)!(t+2)!}\sum_{k=0}^{t+2}(-1)^k\binom{t+2}{k}\mathrm{B}\left(\alpha(s+k+1),1-\frac{1}{\eta}\right)\right],$$
$$=\frac{1}{3}\alpha\frac{(s+t+3)!}{(s+2)!}\left[\sum_{k=0}^{t}\frac{(-1)^k}{(k)!(t-k)!}\mathrm{B}\left(\alpha(s+k+3),1-\frac{1}{\eta}\right)\right.$$
$$-2\frac{(s+2)}{(t+1)!}\left(\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)+\sum_{k=0}^{t}\frac{(-1)^{k+1}(t+1)!}{(k+1)!(t-k)!}\mathrm{B}\left(\alpha(s+k+3),1-\frac{1}{\eta}\right)\right)$$
$$+\frac{(s+2)(s+1)}{(t+2)!}\left(\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)-(t+2)\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)\right.$$
$$\left.\left.+\sum_{k=0}^{t}\frac{(-1)^{k+2}(t+2)!}{(k+2)!(t-k)!}\mathrm{B}\left(\alpha(s+k+3),1-\frac{1}{\eta}\right)\right)\right],$$
$$=\frac{1}{3}\alpha\frac{(s+t+3)!}{(s+2)!}\left[\frac{(s+1)(s+2)}{(t+2)!}\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)-\frac{(s+2)(s+3)}{(t+1)!}\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)\right.$$
$$\left.+\sum_{k=0}^{t}\frac{(-1)^k}{(k+2)!(t-k)!}\mathrm{B}\left(\alpha(s+k+3),1-\frac{1}{\eta}\right)\{(k+1)(k+2)+2(k+2)(s+2)+(s+1)(s+2)\}\right] \qquad (4\text{-}13)$$

Hence, the third TL-moments with generalized trimmed for s and t will be:

$$\lambda_3^{(s,t)} = \frac{1}{3}\alpha\frac{(s+t+3)!}{(s+2)!}\left[\frac{(s+1)(s+2)}{(t+2)!}\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)\right.$$
$$-\frac{(s+2)(s+3)}{(t+1)!}\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)$$
$$\left.+\sum_{k=0}^{t}\frac{(-1)^k}{(k+2)!(t-k)!}\mathrm{B}\left(\alpha(s+k+3),1-\frac{1}{\eta}\right)\{(s+k+3)(s+k+4)\}\right] \tag{4-14}$$

Also, the fourth TL-moments with generalized trimmed for s and t by using the relation (4-4) for $\lambda_r^{(s,t)}$ with the case r = 4 can be obtained as follows:

$$\lambda_4^{(s,t)} = \frac{1}{4}\left[\lambda_1^{(s+3,t)} - 3\lambda_1^{(s+2,t+1)} + 3\lambda_1^{(s+1,t+2)} - \lambda_1^{(s,t+3)}\right], \tag{4-15}$$

and using the formula (4-8) of $\lambda_1^{(s,t)}$, then we will have:

$$\lambda_4^{(s,t)} = \frac{1}{4}\alpha\left[\frac{(s+t+4)!}{(s+3)!(t)!}\sum_{k=0}^{t}(-1)^k\binom{t}{k}\mathrm{B}\left(\alpha(s+k+4),1-\frac{1}{\eta}\right)\right.$$
$$-3\frac{(s+t+4)!}{(s+2)!(t+1)!}\sum_{k=0}^{t+1}(-1)^k\binom{t+1}{k}\mathrm{B}\left(\alpha(s+k+3),1-\frac{1}{\eta}\right)$$
$$+3\frac{(s+t+4)!}{(s+1)!(t+2)!}\sum_{k=0}^{t+2}(-1)^k\binom{t+2}{k}\mathrm{B}\left(\alpha(s+k+2),1-\frac{1}{\eta}\right)$$
$$\left.-\frac{(s+t+3)!}{(s)!(t+3)!}\sum_{k=0}^{t+3}(-1)^k\binom{t+3}{k}\mathrm{B}\left(\alpha(s+k+1),1-\frac{1}{\eta}\right)\right], \tag{4-16}$$

Then, we will have:

$$\lambda_4^{(s,t)} = \frac{1}{4}\alpha\frac{(s+t+4)!}{(s+3)!}\left[\sum_{k=0}^{t}\frac{(-1)^k(k+1)(k+2)(k+3)}{(k+3)!(t-k)!}\mathrm{B}\left(\alpha(s+k+4),1-\frac{1}{\eta}\right)\right.$$
$$-3\frac{(s+3)}{(t+1)!}\left(\mathrm{B}\left(\alpha(s+3),1-\frac{1}{\eta}\right)-\sum_{k=0}^{t}\frac{(-1)^k(t+1)!(k+2)(k+3)}{(k+3)!(t-k)!}\mathrm{B}\left(\alpha(s+k+4),1-\frac{1}{\eta}\right)\right)$$
$$+3\frac{(s+2)(s+3)}{(t+2)!}\left(\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)-(t+2)\mathrm{B}\left(\alpha(s+3),1-\frac{1}{\eta}\right)\right.$$
$$\left.+\sum_{k=0}^{t}\frac{(-1)^{k+2}(t+2)!(k+3)}{(k+3)!(t-k)!}\mathrm{B}\left(\alpha(s+k+4),1-\frac{1}{\eta}\right)\right)$$
$$-\frac{(s+1)(s+2)(s+3)}{(t+3)!}\left(\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)-(t+3)\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)\right.$$
$$\left.\left.+\frac{1}{2}(t+3)(t+2)\mathrm{B}\left(\alpha(s+3),1-\frac{1}{\eta}\right)-\sum_{k=0}^{t}\frac{(-1)^k(t+3)!}{(k+3)!(t-k)!}\mathrm{B}\left(\alpha(s+k+4),1-\frac{1}{\eta}\right)\right)\right], \tag{4-17}$$

So, we will have:

$$\lambda_4^{(s,t)}=\frac{1}{4}\alpha\frac{(s+t+4)!}{(s+3)!}\Bigg[-\frac{(s+3)}{(t+1)!}\left(3+3(s+2)+\frac{1}{2}(s+2)(s+1)\right)\mathrm{B}\left(\alpha(s+3),1-\frac{1}{\eta}\right)$$
$$+\frac{(s+2)(s+3)(s+4)}{(t+2)!}\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)-\frac{(s+1)(s+2)(s+3)}{(t+3)!}\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)$$
$$+\sum_{k=0}^{t}\frac{(-1)^k}{(k+3)!(t-k)!}\mathrm{B}\left(\alpha(s+k+4),1-\frac{1}{\eta}\right)\{(k+1)(k+2)(k+3)+3(k+2)(k+3)(s+3)$$
$$+3(k+3)(s+2)(s+3)+(s+1)(s+2)(s+3)\}\Bigg]. \tag{4-18}$$

Hence, the fourth TL-moments with generalized trimmed for s and t will be:

$$\lambda_4^{(s,t)}=\frac{1}{4}\alpha\frac{(s+t+4)!}{(s+3)!}\Bigg[-\frac{(s+3)}{(t+1)!}\left(3(s+3)+\frac{1}{2}(s+2)(s+1)\right)\mathrm{B}\left(\alpha(s+3),1-\frac{1}{\eta}\right)$$
$$+\frac{(s+2)(s+3)(s+4)}{(t+2)!}\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)-\frac{(s+1)(s+2)(s+3)}{(t+3)!}\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)$$
$$+\sum_{k=0}^{t}\frac{(-1)^k}{(k+3)!(t-k)!}\mathrm{B}\left(\alpha(s+k+4),1-\frac{1}{\eta}\right)\{(s+k+4)(s+k+5)(s+k+6)\}\Bigg]. \tag{4-19}$$

We also can obtain the r^{th} TL-moments for the EPD with generalized trimmed (s, t = 0, 1, 2, 3, …), by using the relation between the first TL-moments with generalized trimmed and the r^{th} TL-moments with generalized trimmed, then we will obtain:

$$\lambda_r^{(s,t)}=\frac{\alpha}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\left(\mathrm{B}(r+s-j,t+j+1)\right)^{-1}\sum_{k=0}^{t+j}(-1)^k\binom{t+j}{k}\mathrm{B}\left(\alpha(r+s+k-j),1-\frac{1}{\eta}\right). \tag{4-20}$$

where $r\geq 2$. Putting r = 2, 3 and 4, we can obtain the equations for the second, third and fourth TL-moments with generalized trimmed for s and t as a special cases from the r^{th} TL-moments. From these results we can also obtain the TL-CV $\tau^{(s,t)}$, TL-skewness $\tau_3^{(s,t)}$ and TL-kurtosis $\tau_4^{(s,t)}$ for the EPD. Using equations (4-8) and (4-20), many special cases can be obtained as follows:

4.2 Special Cases

Now, we will obtain the TL-moments with the first trimmed, the L-moments, the LH-moments and the LL-moments for the EPD as a special cases of the TL-moments with generalized trimmed for s and t (s, t = 0, 1, 2, 3, … .) for the EPD.

<u>4.2.1 When($s=t=1$): The TL-Moments with first trimmed</u>

The first four TL-moments for the EPD with the first trimmed can be obtained by substituting $s=1$, and $t=1$ for $\lambda_1^{(s,t)}$, $\lambda_2^{(s,t)}$, $\lambda_3^{(s,t)}$ and $\lambda_4^{(s,t)}$ in equations (4-8), (4-11), (4-14) and (4-19) respectively, as follows :

$$\lambda_1^{(1)} = \lambda_1^{(1,1)} = \alpha \frac{(3)!}{(1)!(1)!} \sum_{k=0}^{1} (-1)^k \binom{1}{k} \mathrm{B}\left(\alpha(k+2), 1-\frac{1}{\eta} \right) - 1,$$

$$= 6\alpha \left[\mathrm{B}\left(2\alpha, 1-\frac{1}{\eta} \right) - \mathrm{B}\left(3\alpha, 1-\frac{1}{\eta} \right) \right] - 1, \tag{4-21}$$

and the second TL-moments with the first trimmed for the EPD will be:

$$\lambda_2^{(1)} = \lambda_2^{(1,1)} = \frac{1}{2} \alpha \frac{(4)!}{(2)!} \left[-\frac{(2)}{(2)!} \mathrm{B}\left(2\alpha, 1-\frac{1}{\eta} \right) + \sum_{k=0}^{1} \frac{(-1)^k (k+3)}{(k+1)!(1-k)!} \mathrm{B}\left(\alpha(k+3), 1-\frac{1}{\eta} \right) \right],$$

$$= 6\alpha \left[3\mathrm{B}\left(3\alpha, 1-\frac{1}{\eta} \right) - 2\mathrm{B}\left(4\alpha, 1-\frac{1}{\eta} \right) - \mathrm{B}\left(2\alpha, 1-\frac{1}{\eta} \right) \right], \tag{4-22}$$

and the third TL-moments with the first trimmed for the EPD will be:

$$\lambda_3^{(1)} = \lambda_3^{(1,1)} = \frac{1}{3} \alpha \frac{(5)!}{(3)!} \left[\frac{(2)(3)}{(3)!} \mathrm{B}\left(2\alpha, 1-\frac{1}{\eta} \right) - \frac{(3)(4)}{(2)!} \mathrm{B}\left(3\alpha, 1-\frac{1}{\eta} \right) \right.$$

$$\left. + \sum_{k=0}^{1} \frac{(-1)^k}{(k+2)!(1-k)!} \mathrm{B}\left(\alpha(k+4), 1-\frac{1}{\eta} \right) \{(k+4)(k+5)\} \right],$$

$$= \frac{20}{3} \alpha \left[\mathrm{B}\left(2\alpha, 1-\frac{1}{\eta} \right) - 6\mathrm{B}\left(3\alpha, 1-\frac{1}{\eta} \right) + 10\mathrm{B}\left(4\alpha, 1-\frac{1}{\eta} \right) - 5\mathrm{B}\left(5\alpha, 1-\frac{1}{\eta} \right) \right]. \tag{4-23}$$

Then, the fourth TL-moment with the first trimmed for the EPD will be:

$$\lambda_4^{(1)} = \lambda_4^{(1,1)} = \frac{1}{4} \alpha \frac{(6)!}{(4)!} \left[-30\mathrm{B}\left(4\alpha, \frac{\eta-1}{\eta} \right) + 10\mathrm{B}\left(3\alpha, 1-\frac{1}{\eta} \right) \right.$$

$$\left. - \mathrm{B}\left(2\alpha, 1-\frac{1}{\eta} \right) + \sum_{k=0}^{1} \frac{(-1)^k}{(k+3)!(1-k)!} \mathrm{B}\left(\alpha(k+5), 1-\frac{1}{\eta} \right) \{(k+5)(k+6)(k+7)\} \right], \tag{4-24}$$

So, we will have:

$$\lambda_4^{(1)} = \lambda_4^{(1,1)} = \frac{15}{2} \alpha \left[35\mathrm{B}\left(5\alpha, 1-\frac{1}{\eta} \right) - 14\mathrm{B}\left(6\alpha, 1-\frac{1}{\eta} \right) - 30\mathrm{B}\left(4\alpha, 1-\frac{1}{\eta} \right) + 10\mathrm{B}\left(3\alpha, 1-\frac{1}{\eta} \right) \right.$$

$$\left. - \mathrm{B}\left(2\alpha, 1-\frac{1}{\eta} \right) \right]. \tag{4-25}$$

Also, from the TL-coefficient of variation $\tau^{(s,t)}$, the TL-skewness $\tau_3^{(s,t)}$ and the TL-kurtosis $\tau_4^{(s,t)}$ with generalized trimmed for s and t for the EPD, and by putting s = t = 1, we can compute the TL-coefficient of variation $\tau^{(1,1)}$, TL-skewness $\tau_3^{(1,1)}$ and TL-kurtosis $\tau_4^{(1,1)}$ with the first trimmed for the EPD.

Also, from the r^{th} TL-moments with generalized trimmed for the EPD for s and t (s, t = 0, 1, 2, 3, …) and by putting s = t =1, we can obtain the r^{th} TL-moments with the first trimmed for the EPD as follows:

$$\lambda_r^{(1,1)} = \frac{\alpha}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\big(\mathrm{B}(r+1-j,\, j+2)\big)^{-1}\sum_{k=0}^{j+1}(-1)^k\binom{j+1}{k}\mathrm{B}\left(\alpha(r+k+1-j),1-\frac{1}{\eta}\right). \tag{4-26}$$

4.2.2 When($s=t=0$): The L-Moments

Now, we can obtain the first four L-moment for the EPD as a special case from the TL-moments with generalized trimmed for the EPD by substituting $s=0$, and $t=0$ for $\lambda_1^{(s,t)}$, $\lambda_2^{(s,t)}$, $\lambda_3^{(s,t)}$ and $\lambda_4^{(s,t)}$ in equations (4-8), (4-11), (4-14) and (4-19) respectively, we will obtain $\lambda_1^{(0,0)}$, $\lambda_2^{(0,0)}$, $\lambda_3^{(0,0)}$, and $\lambda_3^{(0,0)}$ as follows:

$$\lambda_1 = \alpha\mathrm{B}\left(\alpha,1-\frac{1}{\eta}\right)-1, \tag{4-27}$$

$$\lambda_2 = \alpha\left[2\mathrm{B}\left(2\alpha,1-\frac{1}{\eta}\right)-\mathrm{B}\left(\alpha,1-\frac{1}{\eta}\right)\right], \tag{4-28}$$

$$\lambda_3 = \alpha\left[6\mathrm{B}\left(3\alpha,1-\frac{1}{\eta}\right)-6\mathrm{B}\left(2\alpha,1-\frac{1}{\eta}\right)+\mathrm{B}\left(\alpha,1-\frac{1}{\eta}\right)\right], \tag{4-29}$$

and

$$\lambda_4 = \alpha\left[20\mathrm{B}\left(4\alpha,1-\frac{1}{\eta}\right)-30\mathrm{B}\left(3\alpha,1-\frac{1}{\eta}\right)+12\mathrm{B}\left(2\alpha,1-\frac{1}{\eta}\right)-\mathrm{B}\left(\alpha,1-\frac{1}{\eta}\right)\right]. \tag{4-30}$$

The results for the first two L-moments in equations (4-27) and (4-28) are the same as the results that obtained by Shawky and Abu-Zinadah [2].

Also, from the TL-coefficient of variation $\tau^{(s,t)}$, the TL-skewness $\tau_3^{(s,t)}$ and the TL-kurtosis $\tau_4^{(s,t)}$ with generalized trimmed for s and t for the EPD, and by putting s = t =0, we can compute the L-coefficient of variation τ , L-skewness τ_3 and L-kurtosis τ_4 for the EPD.

From the rth TL-moments with generalized trimmed for the EPD for s and t (s, t = 0, 1, 2, 3, …)and by putting s = t = 0, we can obtain the rth L-moments for the EPD as follows:

$$\lambda_r = \lambda_r^{(0,0)} = \frac{\alpha}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\big(\mathrm{B}(r-j,\, j+1)\big)^{-1}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\mathrm{B}\left(\alpha(r+k-j),1-\frac{1}{\eta}\right). \tag{4-31}$$

where $r \geq 2$, and by putting r = 2, 3, 4, we can obtain the second, the third and the fourth L-moments for the EPD as special cases.

4.2.3 When($t = 0$): The LH-Moments

By taking $t = 0$ for the TL-moments with generalized trimmed for the EPD, we will obtain the LH-moments with generalized trimmed for s for the EPD. The LH-moments are linear functions of the expectations of the highest order statistic and introduced by Wang [10] as a modified version of L-moments, to characterize the upper part of a distribution. When one wants to put more emphasis on extreme events, LH-moment approach allows to giving more weight to the largest ones. When $s = 0$ corresponds to the L-moments. As s increases, LH-moments reflect more and more the characteristics of the upper part of the data. Wang [10] found that the method of LH-moments resulted in large sampling variability for high s, and recommended not to use values of s higher than 4.

By substituting $t = 0$, for $\lambda_1^{(s,t)}$, $\lambda_2^{(s,t)}$, $\lambda_3^{(s,t)}$ and $\lambda_4^{(s,t)}$ in equations (4-8), (4-11), (4-14) and (4-19) respectively, then we will obtain the first four LH-moments with generalized trimmed for s for the EPD as follows:

$$\lambda_1^{(s,0)} = \alpha(s+1)\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)-1. \tag{4-32}$$

$$\lambda_2^{(s,0)} = \frac{(s+2)}{2}\alpha\left[-(s+1)\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)+(s+2)\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)\right]. \tag{4-33}$$

$$\lambda_3^{(s,0)} = \frac{(s+3)}{3}\alpha\left[\frac{1}{2}(s+1)(s+2)\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)\right.$$
$$\left.-(s+2)(s+3)\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)+\frac{1}{2}(s+3)(s+4)\mathrm{B}\left(\alpha(s+3),1-\frac{1}{\eta}\right)\right] \tag{4-34}$$

and

$$\lambda_4^{(s,0)} = \frac{(s+4)}{4}\alpha\left[-(s+3)\left(3(s+3)+\frac{1}{2}(s+2)(s+1)\right)\mathrm{B}\left(\alpha(s+3),1-\frac{1}{\eta}\right)\right.$$
$$+\frac{1}{2}(s+2)(s+3)(s+4)\mathrm{B}\left(\alpha(s+2),1-\frac{1}{\eta}\right)-\frac{1}{6}(s+1)(s+2)(s+3)\mathrm{B}\left(\alpha(s+1),1-\frac{1}{\eta}\right)$$
$$\left.+\frac{1}{6}(s+4)(s+5)(s+6)\mathrm{B}\left(\alpha(s+4),1-\frac{1}{\eta}\right)\right]. \tag{4-35}$$

Hence, from these results we can also obtain the LH-ratios as the LH-coefficient of variation $\tau^{(s,0)}$, LH-skewness $\tau_3^{(s,0)}$ and LH-kurtosis $\tau_4^{(s,0)}$ with generalized trimmed for s for the EPD.By putting t = 0 in the rth TL-moments with generalized trimmed, we can obtain the rth LH-moments for the EPD with generalized trimmed for s (s = 0, 1, 2, 3, …)as follows:

$$\lambda_r^{(s,0)} = \frac{\alpha}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\left(\mathrm{B}(r+s-j,\, j+1)\right)^{-1}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\mathrm{B}\left(\alpha(r+s+k-j),1-\frac{1}{\eta}\right). \tag{4-36}$$

where $r \geq 2$, and by putting r = 2, 3, 4, we can obtain the equations for the second, third and fourth LH-moments for the EPD as a special cases from the r[th] LH-moments for the EPD. Also, we can obtain the LH-moments with the first trimmed $s = 1$, second trimmed $s = 2$, third trimmed $s = 3$ and the fourth trimmed $s = 4$ as a special cases from the LH-moments with generalized trimmed for s for the EPD.

4.2.4 When($s = 0$): The LL-Moments

By taking $s = 0$ for the TL-moments with generalized trimmed for the EPD, we will obtain the LL-moments with generalized trimmed for t for the EPD. The LL-moments are linear functions of the expectations of the lowest order statistic and introduced by Bayazit and Önöz [11]. L-moments are a special case for $s = 0$. As t increases, increases the weight of the lower part of the data.

By substituting $s = 0$, for $\lambda_1^{(s,t)}$, $\lambda_2^{(s,t)}$, $\lambda_3^{(s,t)}$ and $\lambda_4^{(s,t)}$ in equations (4-8), (4-11), (4-14) and (4-19) respectively, then we will obtain the first four LL-moments with generalized trimmed for t for the EPD as follows:

$$\lambda_1^{(0,t)} = (t+1)\alpha \sum_{k=0}^{t} (-1)^k \binom{t}{k} \mathrm{B}\left(\alpha(k+1), 1-\frac{1}{\eta}\right) - 1, \tag{4-37}$$

$$\lambda_2^{(0,t)} = \frac{(t+2)!}{2}\alpha\left[-\frac{1}{(t+1)!}\mathrm{B}\left(\alpha, 1-\frac{1}{\eta}\right) + \sum_{k=0}^{t}\frac{(-1)^k (k+2)}{(k+1)!(t-k)!}\mathrm{B}\left(\alpha(k+2), 1-\frac{1}{\eta}\right)\right], \tag{4-38}$$

$$\begin{aligned}\lambda_3^{(0,t)} = \frac{(t+3)!}{6}\alpha\Bigg[&\frac{2}{(t+2)!}\mathrm{B}\left(\alpha, 1-\frac{1}{\eta}\right) - \frac{6}{(t+1)!}\mathrm{B}\left(2\alpha, 1-\frac{1}{\eta}\right)\\ &+ \sum_{k=0}^{t}\frac{(-1)^k (k+3)(k+4)}{(k+2)!(t-k)!}\mathrm{B}\left(\alpha(k+3), 1-\frac{1}{\eta}\right)\Bigg],\end{aligned} \tag{4-39}$$

and

$$\begin{aligned}\lambda_4^{(0,t)} = \frac{(t+4)!}{24}\alpha\Bigg[&-\frac{30}{(t+1)!}\mathrm{B}\left(3\alpha, 1-\frac{1}{\eta}\right) + \frac{24}{(t+2)!}\mathrm{B}\left(2\alpha, 1-\frac{1}{\eta}\right) - \frac{6}{(t+3)!}\mathrm{B}\left(\alpha, 1-\frac{1}{\eta}\right)\\ &+ \sum_{k=0}^{t}\frac{(-1)^k (k+4)(k+5)(k+6)}{(k+3)!(t-k)!}\mathrm{B}\left(\alpha(k+4), 1-\frac{1}{\eta}\right)\Bigg].\end{aligned} \tag{4-40}$$

We also can obtain the r[th] LL-moments for the EPD with generalized trimmed t (t = 0, 1, 2, 3, …), by putting s = 0 in the r[th] TL-moments with generalized trimmed, then we will have:

$$\lambda_r^{(0,t)} = \frac{\alpha}{r}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \left(\mathrm{B}(r-j, t+j+1)\right)^{-1} \sum_{k=0}^{t+j} (-1)^k \binom{t+j}{k} \mathrm{B}\left(\alpha(r+k-j), 1-\frac{1}{\eta}\right). \tag{4-41}$$

where $r \geq 2$. Hence, from these results we can compute the LL-coefficient of variation $\tau^{(0,t)}$, LL-skewness $\tau_3^{(0,t)}$ and LL-kurtosis $\tau_4^{(0,t)}$ with generalized trimmed for t for the EPD. Also, for t (t = 0, 1, 2, 3, …), we can obtain the LL-moments with any trimmed for the EPD.

4.2.5 When($\alpha=1$): Results for the Standard Pareto Distribution

By putting $\alpha=1$, we can obtain the rth TL-moments for the standard Pareto distribution as a special case from the rth TL-moments with generalized trimmed for s and t for the EPD and from the rth TL-moments, we can obtain the first four TL-moments with generalized trimmed (s, t = 0, 1, 2, …) for the standard pareto distribution and so, we can obtain the TL-ratios.

Let, $\alpha=1$ in equation (4-8) and (4-20), the first TL-moments for the standard Pareto distribution can be obtained as a special case from the first TL-moments for the EPD as follows:

$$\lambda_1^{(s,t)}=\frac{(s+t+1)!}{(s)!(t)!}\sum_{k=0}^{t}(-1)^k\binom{t}{k}\mathrm{B}\left(s+k+1,1-\frac{1}{\eta}\right)-1. \tag{4-42}$$

and the rth TL-moments will be:

$$\lambda_r^{(s,t)}=\frac{1}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\left(\mathrm{B}(r+s-j,t+j+1)\right)^{-1}\sum_{k=0}^{t+j}(-1)^k\binom{t+j}{k}\mathrm{B}\left(r+s+k-j,1-\frac{1}{\eta}\right). \tag{4-43}$$

Where $r\geq 2$. And by taking s = t = 0 with $\alpha=1$, for the rth TL-moments for the EPD, we will obtain the rth L-moments for the standard Pareto distribution and from these we can obtain the first four L-moments and L-ratios for the standard Pareto distribution. Also, by taking t = 0 in the rth TL-moments for the standard Pareto distribution, we will obtain the rth LH-moments for the standard Pareto distribution and from these we can obtain the first four LH-moments and the LH-ratios and by taking s = 0, we will obtain the rth LL-moments for the standard Pareto distribution and from these we can obtain the first four LL-moments and the LL-ratios for the standard Pareto distribution.

4.3 TL-moments Estimators (TLMEs) for the EPD

The TL-moment estimators (TLMEs) for the unknown parameters of the EPD can be obtained by equating the first two population TL-moments ($\lambda_1^{(s,t)}$, $\lambda_2^{(s,t)}$) to the corresponding sample TL-moments ($l_1^{(s,t)}$, $l_2^{(s,t)}$) for the EPD. Hosking [9] obtained the first two sample TL-moments as follows:

$$l_1^{(s,t)}=\frac{1}{\binom{n}{s+t+1}}\sum_{j=s+1}^{n-t}\binom{j-1}{s}\binom{n-j}{t}x_{(j:n)}, \tag{4-44}$$

and

$$l_2^{(s,t)}=\frac{1}{2\binom{n}{s+t+2}}\sum_{j=s+1}^{n-t}\binom{j-1}{s}\binom{n-j}{t}\left(\frac{(j-s-1)}{(s+1)}-\frac{(n-j-t)}{(t+1)}\right)x_{(j:n)} \tag{4-45}$$

Now, the TL-moment estimators (TLMEs) $\hat{\alpha}$ and $\hat{\eta}$ of the EPD will be obtained by solving the following two equations:

$$l_1^{(s,t)} = \frac{(s+t+1)!}{(s)!(t)!}\hat{\alpha}\sum_{k=0}^{t}(-1)^k\binom{t}{k}\mathrm{B}\left(\hat{\alpha}(s+k+1),1-\frac{1}{\hat{\eta}}\right)-1. \tag{4-46}$$

and

$$l_2^{(s,t)} = \frac{1}{2}\hat{\alpha}\frac{(s+t+2)!}{(s+1)!}\left[-\frac{(s+1)}{(t+1)!}\mathrm{B}\left(\hat{\alpha}(s+1),1-\frac{1}{\hat{\eta}}\right)+\sum_{k=0}^{t}\frac{(-1)^k(s+k+2)}{(k+1)!(t-k)!}\mathrm{B}\left(\hat{\alpha}(s+k+2),1-\frac{1}{\hat{\eta}}\right)\right]. \tag{4-47}$$

Equations (4-46) and (4-47) are valid for any trimmed s and t. Since, beta function is a function of two shape parameters α and η, these equations will be solved numerically. As a special case, the L-moments estimators $\hat{\alpha}$ and $\hat{\eta}$ for the EPD will be obtained by putting s = t = 0 which are the same as Shawky and Abu-Zinadah [2] results for the EPD. Also, as a special case, the TLMEs $\hat{\alpha}$ and $\hat{\eta}$ with the first trimmed for the EPD may be obtained by putting s = t = 1. For $\alpha = 1$, and by putting s = t= 1, the TLME $\hat{\eta}$ with the first trimmed for the standard Pareto distribution can be obtained.

5. LQ-MOMENTS ESTIMATORS (LQMEs) FOR THE EPD

In this section, LQ-moments with the three different cases (median, trimean and Gastwirth) for the EPD will be obtained and used to estimate the unknown parameters of the EPD.

5.1 LQ-Moments for the EPD

Let, $X_1, X_2, ..., X_n$ be a random sample from a continuous distribution function $F(x)$ with quantile function $Q_X(u) = F_X^{-1}(u)$ and let, $X_{(1:n)} \le X_{(2:n)} \le ... \le X_{(n:n)}$ denote the order statistics. Mudholkar and Hutson [5] defined the rth population LQ-moments ζ_r of X, as:

$$\zeta_r = r^{-1}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\tau_{p,d}(X_{(r-k:r)}), \qquad r = 1,2,... \tag{5-1}$$

where $0 \le d \le 1/2,$ $0 \le p \le 1/2,$ and

$$\tau_{p,d}(X_{(r-k:r)}) = pQ_{X_{(r-k:r)}}(d) + (1-2p)Q_{X_{(r-k:r)}}(1/2) + pQ_{X_{(r-k:r)}}(1-d) \tag{5-2}$$

The linear combination $\tau_{p,d}$ is a 'quick' measure of the location of the sampling distribution of the order statistic $X_{(r-j:r)}$. The candidates for $\tau_{p,d}$ include the function generating the common quick estimators by using the median ($p = 0.5,\ d = 0.5$), the trimean ($p = 1/4,\ d = 1/4$) and the Gastwirth ($p = 0.3,\ d = 1/3$). They introduced the LQ-skewness and LQ-kurtosis for the population by $\eta_3 = \zeta_3/\zeta_2$ and $\eta_4 = \zeta_4/\zeta_2$ respectively; it may be used for identifying the population and estimating the parameters. The LQ-skewness takes the value of zero for symmetrical distributions.

The LQ-moments with the three cases (median, trimean and Gastwirth) will be obtained for the EPD as follows:

5.1.1 Using the median case ($p = 0.5,\ d = 0.5$)

By using the quantile function for the EPD, the first four LQ-moments for the EPD will be:

$$\xi_1 = \left[\left(1-(0.5)^{1/\alpha}\right)^{-1/\eta}\right] - 1, \tag{5-3}$$

$$\xi_2 = \frac{1}{2}\left[\left(1-(0.707)^{1/\alpha}\right)^{-1/\eta} - \left(1-(0.293)^{1/\alpha}\right)^{-1/\eta}\right], \tag{5-4}$$

$$\xi_3 = \frac{1}{3}\left[\left(1-(0.794)^{1/\alpha}\right)^{-1/\eta} - 2\left(1-(0.5)^{1/\alpha}\right)^{-1/\eta} + \left(1-(0.206)^{1/\alpha}\right)^{-1/\eta}\right], \tag{5-5}$$

and

$$\xi_4 = \frac{1}{4}\left[\left(1-(0.841)^{1/\alpha}\right)^{-1/\eta} - 3\left(1-(0.614)^{1/\alpha}\right)^{-1/\eta} + 3\left(1-(0.386)^{1/\alpha}\right)^{-1/\eta} - \left(1-(0.159)^{1/\alpha}\right)^{-1/\eta}\right] \tag{5-6}$$

5.1.2 Using the trimean case ($p = 1/4,\ d = 1/4$)

By using the quantile function for the EPD, the first four LQ-moments for the EPD will be:

$$\xi_1 = \frac{1}{4}\left[\left(1-(0.25)^{1/\alpha}\right)^{-1/\eta} + 2\left(1-(0.5)^{1/\alpha}\right)^{-1/\eta} + \left(1-(0.75)^{1/\alpha}\right)^{-1/\eta}\right] - 1, \tag{5-7}$$

$$\xi_2 = \frac{1}{8}\left[2\left(1-(0.707)^{1/\alpha}\right)^{-1/\eta} - 2\left(1-(0.293)^{1/\alpha}\right)^{-1/\eta} + \left(1-(0.866)^{1/\alpha}\right)^{-1/\eta} - \left(1-(0.134)^{1/\alpha}\right)^{-1/\eta}\right], \tag{5-8}$$

$$\begin{aligned}\xi_3 = \frac{1}{12}\Big[&\left(1-(0.909)^{1/\alpha}\right)^{-1/\eta} + 2\left(1-(0.794)^{1/\alpha}\right)^{-1/\eta} - 2\left(1-(0.674)^{1/\alpha}\right)^{-1/\eta} + \left(1-(0.630)^{1/\alpha}\right)^{-1/\eta} \\ &- 4\left(1-(0.5)^{1/\alpha}\right)^{-1/\eta} + \left(1-(0.370)^{1/\alpha}\right)^{-1/\eta} - 2\left(1-(0.326)^{1/\alpha}\right)^{-1/\eta} + 2\left(1-(0.206)^{1/\alpha}\right)^{-1/\eta} \\ &+ \left(1-(0.091)^{1/\alpha}\right)^{-1/\eta}\Big],\end{aligned} \tag{5-9}$$

and

$$\begin{aligned}\xi_4 = \frac{1}{16}\Big[&\left(1-(0.931)^{1/\alpha}\right)^{-1/\eta} + 2\left(1-(0.841)^{1/\alpha}\right)^{-1/\eta} - 3\left(1-(0.757)^{1/\alpha}\right)^{-1/\eta} + \left(1-(0.707)^{1/\alpha}\right)^{-1/\eta} \\ &- 6\left(1-(0.614)^{1/\alpha}\right)^{-1/\eta} + 3\left(1-(0.544)^{1/\alpha}\right)^{-1/\eta} - 3\left(1-(0.456)^{1/\alpha}\right)^{-1/\eta} + 6\left(1-(0.386)^{1/\alpha}\right)^{-1/\eta} \\ &- \left(1-(0.293)^{1/\alpha}\right)^{-1/\eta} + 3\left(1-(0.243)^{1/\alpha}\right)^{-1/\eta} - 2\left(1-(0.159)^{1/\alpha}\right)^{-1/\eta} - \left(1-(0.069)^{1/\alpha}\right)^{-1/\eta}\Big]\end{aligned} \tag{5-10}$$

5.1.3 Using the Gastwirth case ($p = 0.3,\ d = 1/3$)

By using the quantile function for the EPD, the first four LQ-moments for the EPD will be:

$$\xi_1 = \frac{1}{10}\left[3\left(1-(0.333)^{1/\alpha}\right)^{-1/\eta} + 4\left(1-(0.5)^{1/\alpha}\right)^{-1/\eta} + 3\left(1-(0.667)^{1/\alpha}\right)^{-1/\eta}\right] - 1, \tag{5-11}$$

$$\xi_2 = \frac{1}{20}\Big[3\left(1-(0.816)^{1/\alpha}\right)^{-1/\eta} + 4\left(1-(0.707)^{1/\alpha}\right)^{-1/\eta} + 3\left(1-(0.577)^{1/\alpha}\right)^{-1/\eta} - 3\left(1-(0.423)^{1/\alpha}\right)^{-1/\eta} - 8\left(1-(0.293)^{1/\alpha}\right)^{-1/\eta} - 3\left(1-(0.184)^{1/\alpha}\right)^{-1/\eta}\Big] \quad (5\text{-}12)$$

$$\xi_3 = \frac{1}{30}\Big[3\left(1-(0.874)^{1/\alpha}\right)^{-1/\eta} + 4\left(1-(0.794)^{1/\alpha}\right)^{-1/\eta} + 3\left(1-(0.693)^{1/\alpha}\right)^{-1/\eta} - 6\left(1-(0.613)^{1/\alpha}\right)^{-1/\eta} - 8\left(1-(0.5)^{1/\alpha}\right)^{-1/\eta} - 6\left(1-(0.387)^{1/\alpha}\right)^{-1/\eta} + 3\left(1-(0.307)^{1/\alpha}\right)^{-1/\eta} + 4\left(1-(0.206)^{1/\alpha}\right)^{-1/\eta} + 3\left(1-(0.126)^{1/\alpha}\right)^{-1/\eta}\Big], \quad (5\text{-}13)$$

and

$$\xi_4 = \frac{1}{40}\Big[3\left(1-(0.904)^{1/\alpha}\right)^{-1/\eta} + 4\left(1-(0.841)^{1/\alpha}\right)^{-1/\eta} + 3\left(1-(0.760)^{1/\alpha}\right)^{-1/\eta} - 9\left(1-(0.709)^{1/\alpha}\right)^{-1/\eta} - 12\left(1-(0.614)^{1/\alpha}\right)^{-1/\eta} + 9\left(1-(0.514)^{1/\alpha}\right)^{-1/\eta} - 9\left(1-(0.486)^{1/\alpha}\right)^{-1/\eta} + 12\left(1-(0.386)^{1/\alpha}\right)^{-1/\eta} + 9\left(1-(0.291)^{1/\alpha}\right)^{-1/\eta} - 3\left(1-(0.240)^{1/\alpha}\right)^{-1/\eta} - 4\left(1-(0.159)^{1/\alpha}\right)^{-1/\eta} - 3\left(1-(0.096)^{1/\alpha}\right)^{-1/\eta}\Big] \quad (5\text{-}14)$$

Then, the LQ-skewness and the LQ-kurtosis for each case (median, trimean and Gastwirth) for the EPD can be obtained by using the results for the first four LQ-moments for the EPD.

5.2 LQ-Moments Estimators (LQMEs) For the EPD

To estimate the unknown parameters α and η for the EPD using the LQ-moments, the first and the second sample LQ-moments for the EPD will be obtained by using the following definition of the r[th] sample LQ-moments:

$$\hat{\zeta}_r = r^{-1}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\hat{\tau}_{p,d}(X_{(r-k:r)}), \qquad r=1,2,... \quad (5\text{-}15)$$

where

$$\hat{\tau}_{p,d}(X_{(r-k:r)}) = p\hat{Q}_{X_{(r-k:r)}}(d) + (1-2p)\hat{Q}_{X_{(r-k:r)}}(1/2) + p\hat{Q}_{X_{(r-k:r)}}(1-d). \quad (5\text{-}16)$$

$\hat{\tau}_{p,d}(X_{(r-k:r)})$ is the quick estimator of the location for the distribution of $X_{(r-k:r)}$ in a random sample of size r, and $\hat{Q}_X(.)$ denotes the linear interpolation estimator of Q(u) given by:

$$\hat{Q}_X(u) = (1-\varepsilon)X_{[n'u]:n} + \varepsilon X_{[n'u]+1:n}, \quad (5\text{-}17)$$

where $\varepsilon = n'u - [n'u]$, $n' = n+1$ and $[n'u]$ denotes the integral part of nu. Then, the first two sample LQ-moments are given by:

$$\hat{\zeta}_1 = \hat{\tau}_{p,\alpha}(X_{(1:1)}), \quad (5\text{-}18)$$

and

$$\hat{\xi}_2 = \frac{1}{2}\left[\hat{\tau}_{p,\alpha}(X_{(2:2)}) - \hat{\tau}_{p,\alpha}(X_{(1:2)})\right]. \tag{5-19}$$

By equating the first two population LQ-moments with the first two sample LQ-moments for the EPD for each case (median, trimean, and Gastwirth), the LQ-moments estimators for the two unknown parameters will be obtained for each case. Now, the unknown parameters α and η for the EPD using the LQ-moments with the median case (LQMEm) will be estimated by solving the following equations:

$$\hat{\hat{\xi}}_1 = \left[\hat{Q}_\circ(0.5)\right] - 1, \tag{5-20}$$

and

$$\hat{\hat{\xi}}_2 = \frac{1}{2}\left[\hat{Q}_\circ(0.707) - \hat{Q}_\circ(0.293)\right], \tag{5-21}$$

where

$$\hat{Q}_\circ(u) = \left(1 - u^{1/\hat{\hat{\alpha}}}\right)^{-1/\hat{\hat{\eta}}} - 1. \tag{5-22}$$

For the trimean case, the LQ-moments estimates (LQMEt) $\hat{\hat{\alpha}}$ and $\hat{\hat{\eta}}$ will be obtained by solving the following equations:

$$\hat{\hat{\xi}}_1 = \frac{1}{4}\left[\hat{Q}_\circ(0.25) + 2\hat{Q}_\circ(0.5) + \hat{Q}_\circ(0.75)\right] - 1, \tag{5-23}$$

and

$$\hat{\hat{\xi}}_2 = \frac{1}{8}\left[2\hat{Q}_\circ(0.707) - 2\hat{Q}_\circ(0.293) + \hat{Q}_\circ(0.866) - \hat{Q}_\circ(0.134)\right]. \tag{5-24}$$

For the Gastwirth case, the LQ-moments estimates (LQMEg) $\hat{\hat{\alpha}}$ and $\hat{\hat{\eta}}$ will be obtained by solving the following equations:

$$\hat{\hat{\xi}}_1 = \frac{1}{10}\left[3\hat{Q}_\circ(0.333) + 4\hat{Q}_\circ(0.5) + 3\hat{Q}_\circ(0.667)\right] - 1, \tag{5-25}$$

and

$$\hat{\hat{\xi}}_2 = \frac{1}{20}\left[3\hat{Q}_\circ(0.816) + 4\hat{Q}_\circ(0.707) + 3\hat{Q}_\circ(0.577) - 3\hat{Q}_\circ(0.423) - 4\hat{Q}_\circ(0.293) - 3\hat{Q}_\circ(0.184)\right]. \tag{5-26}$$

As a special case, if $\alpha = 1$, the unknown parameter η will be estimated for the standard Pareto distribution using the LQ-moments. To obtain the LQ-moments estimator $\hat{\hat{\eta}}$, solving the equation for $\hat{\hat{\xi}}_1$ numerically for the standard Pareto distribution.

6. A SIMULATION STUDY OF THE EPD

A simulation study will be introduced to compare the performances of seven different estimators: maximum likelihood estimators (MLEs), method of moment estimators (MMEs), L-moments estimators (LMEs), TL-moments estimators (TLMEs) and the three LQ-moments estimators for the three different cases [median (LQMEm), trimean (LQMEt) and Gastwirth (LQMEg)] for the unknown parameters of the EPD. The comparison will be mainly based on their biases and root mean squared errors (RMSEs). The simulation experiments are performed using the Mathcad (14) software, different sample sizes n = 10, 30 and 50, and different values for the shape parameter $\alpha = 0.5, 1.0$ and 3.0 and $\eta = 3$. For each combination of the sample size and the shape parameters values, the experiment will be repeated 10,000 times. In each experiment, the biases and RMSEs for the estimates of α and η will be obtained and listed in Table (1) and (2).

7. RESULTS AND CONCLUSION

It is observed that the biases and RMSEs of the different estimators of α and η depend on the value of the shape parameter α (as α increases the biases and RMSEs increase (see Table (1)). On the other hand, the biases and RMSEs of the estimators of η decrease as α increases for all methods (see Table (2)) except for the MME (increase). For all cases, the biases and RMSEs of the different estimators of α and η decrease as sample size increases.

From (Table 1), it can be seen that most of the estimators are positively biased, this indicates that most of the methods overestimate α except for MME, where the method underestimates α when $\alpha = 0.5$ and n = 50. Comparing the biases of different estimators of α, it is clear that the method of LQMEt yields the minimum bias in almost all cases considered for estimating α. Considering the RMSEs of the different estimators of α, it is clear that the estimator have the minimum RMSEs for small sample size (n = 10) is the LME, and the MLE (TLME is more close to MLE) for large sample size.

From (Table 2), by comparing the biases of different estimators of η, it is clear that the method of LQMEg and LQMEt yield the minimum bias in almost all cases considered for estimating η. Considering the RMSEs of the different estimators of η, it is clear that the estimator have the minimum RMSEs is the MME (LQMEt is more close to MME), except when $\alpha = 3$ for n = 30 and 50 (MLE).

Table 1. Biases and RMSEs of the parameter estimatorsfor different types of Estimators for α

n	α		MLE	MME	LME	TLME	LQMEm	LQMEt	LQMEg
10	0.5	Bias	0.16219	0.09738	0.15322	0.16741	0.30339	0.04504*	0.14721
		RMSEs	0.42785*	1.65547	0.57901	0.93265	2.06768	0.54777	0.87889
	1	Bias	0.43797	0.39387	0.41133	0.54533	1.34588	0.09513*	0.34748
		RMSEs	1.36990	1.71575	1.26170*	2.22309	25.81104	1.61855	2.58212
	3	Bias	2.75322	3.94666	2.34675*	2.70429	3.08388	2.58646	2.90643
		RMSEs	21.34106	8.66075	7.22688*	8.06349	27.99000	23.99594	21.47583
30	0.5	Bias	0.03980	-0.01036*	0.05148	0.02811	0.08068	0.03076	0.03319
		RMSEs	0.13412*	0.23816	0.19149	0.17775	0.31117	0.20593	0.20611
	1	Bias	0.10360	0.13696	0.13218	0.09592	0.20355	0.05816*	0.07752
		RMSEs	0.31539*	0.51715	0.44034	0.44045	0.83288	0.45062	0.48871
	3	Bias	0.47390	2.05877	0.58325	0.44669	1.14465	0.28993*	0.32826
		RMSEs	1.35975*	2.82931	1.75961	1.80699	5.07045	2.01536	2.23890
50	0.5	Bias	0.02488	-0.03050	0.03735	0.01876*	0.04772	0.02076	0.02153
		RMSEs	0.09467*	0.15817	0.14637	0.12816	0.21283	0.15402	0.15230
	1	Bias	0.05374	0.08842	0.08166	0.05308	0.10451	0.03172*	0.04715
		RMSEs	0.21370*	0.32618	0.30362	0.28634	0.49856	0.32676	0.34818
	3	Bias	0.24269	1.73710	0.37710	0.27037	0.55135	0.16246*	0.21473
		RMSEs	0.85929*	2.22061	1.20961	1.16154	2.35047	1.27926	1.41299

**: The least biased value or the least root mean squared errors*

Table 2. Biases and RMSEs of the parameter estimators for different types of Estimators for η

n	α		MLE	MME	LME	TLME	LQMEm	LQMEt	LQMEg
10	0.5	Bias	1.35703	0.12075*	1.31193	1.36834	1.31477	-0.15435	0.57616
		RMSEs	3.16399	0.31467*	3.10277	3.98730	5.56854	2.33895	3.34210
	1	Bias	0.91537	0.37564	0.88814	0.89269	0.57578	-0.25659	0.20028*
		RMSEs	2.09403	0.51654*	2.06783	2.53510	2.94179	1.69342	2.12387
	3	Bias	0.62313	0.80033	0.62267	0.50269*	0.28800	-0.29263	0.04819
		RMSEs	1.46533	0.90798*	1.49749	1.77815	2.06737	1.29886	1.51195
30	0.5	Bias	0.35391	-0.05697*	0.45036	0.31724	0.43029	0.06779	0.12725
		RMSEs	1.11678	0.21263*	1.24428	1.25377	2.11674	1.39515	1.48544
	1	Bias	0.25454	0.17889	0.33503	0.23671	0.21880	-0.00692*	0.04290
		RMSEs	0.85113	0.30604*	0.98660	0.97594	1.47889	0.99805	1.08783
	3	Bias	0.17896	0.57845	0.22555	0.13552	0.12865	-0.03614	-0.02406*
		RMSEs	0.64893*	0.65179	0.76381	0.78213	1.09864	0.77465	0.83157
50	0.5	Bias	0.21594	-0.10618	0.30564	0.19639	0.25089	0.04608*	0.07574
		RMSEs	0.79535	0.21790*	0.93668	0.89612	1.51765	1.02708	1.09073
	1	Bias	0.13653	0.11450	0.21190	0.13820	0.11168	-0.01403*	0.02767
		RMSEs	0.61147	0.24516*	0.73476	0.69733	1.08774	0.77194	0.82453
	3	Bias	0.09852	0.50415	0.15328	0.09106	0.07285	-0.01483	-0.00312*
		RMSEs	0.47190*	0.57042	0.59068	0.56005	0.82720	0.59665	0.64384

**: The least biased value or the least root mean squared errors*

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Gupta RC, Gupta PL, Gupta RD. Modeling failure time data with Lehmann alternative, Communications in Statistics: Theory & Methods. 1998;27:887-904.
2. Shawky AI, Abu-Zinadah HH. Exponentiated pareto distribution: Different Methods of Estimations. Int. J. Contemp. Math. Sciences. 2009;14(4):677–693.
3. Hosking JRM. L-moments: Analysis and estimation of distributions using linear combinations of order statistics. Journal of Royal Statistical Society Series B. 1990;52:105-124.
4. Elamir EAH, Seheult AH. Trimmed l-moments. Computational Statistics & Data Analysis. 2003;43:299–314.
5. Mudholkar GS, Hutson AD. LQ-moments: Analogs of l-moments. Journal of Statistical Planning and Inference. 1998;71:191–208.
6. Abu El-Magd NAT. TL-Moments of the exponentiated generalized extreme value distribution. The Journal of Advanced Research. 2010;1:351–359.
7. Zaher HM, El-Sheik AA, Abu El-Magd NAT. Estimation of pareto parameters using a fuzzy least-squares method and other known techniques with a comparison. British Journal of Mathematics and Computer Science. 2014;4(14):2067-2088.
8. Hosking JRM. Some theory and practical uses of trimmed l-moments. Journal of Statistical Planning and Inference. 2007;137:3024-3039.
9. Maillet B, M´edecin J. Extreme volatilities and l-moment estimations of tail indexes. A conceptual framework. Social Science Research Network. Accessed 12 January 2009. Available: http://ssrn.com/abstract=1288661.
10. Wang Q. LH-moments for statistical analysis of extreme events. Water Resources Research. 1997;33(12):2841-2848.
11. Bayazit M, Önöz B. LL-moments for estimating low flow quantiles. Hydrological Sciences. 2002;47(5):707-720.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sciencedomain.org/review-history.php?iid=743&id=22&aid=6604