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Study of Numerical Stability for Neutral Differential Equations with Delay by Θ- Method

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

This paper discusses numerical asymptotic stability in certain neutral delay differential Equations (NDDEs) by θ -method discretization for $\theta \in [0, 1]$. We give necessary and sufficient conditions on the parameters, to obtain the numerical asymptotic stability, preserving the exact asymptotic stability conditions.

Keywords: Neutral delay differential equation; θ-method discretization; characteristic equation; asymptotic stability.

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1 INTRODUCTION

The Delay Differential Equations (DDE) are a class of infinite dimension systems, widely used for modeling and analysis of phenomenon transmission and propagation (of matter, energy or information). They are still used in the modeling of processes encountered in physics, mechanics, economics, chemistry, biology, dynamics of populations, ecology, physiology and epidemiology [1] [2] [3] [4] [5]. Neutral Delay Differential Equations (NDDEs) is a natural generalization of DDE and, also there is a wide classes of Partial Differential Equations witch can be transformed as a NDDEs (for example [6] and the references therein). For some ordinary differential equations, delay differential equations and integro-differential delay equations, it has been proved that some classical numerical methods can preserve the stability. However, to the best of our knowledge, this is not the case for NDDEs (for example [7]).

In this paper, we consider the following delay differential equation of neutral type

$$x'(t) - cx'(t-\tau) = ax(t) + bx(t-\tau) \qquad (1.1)$$

in which $\tau > 0$ is the delay parameter, c, a and b are real scalars.

From the exact stability conditions, using the relevant result [7] (Theorem 2.1) we may conclude that Equation (1.1) is asymptotically stable if and only if:

$$a \leq b < -a, \qquad |c| < 1, \qquad (1)$$

$$a + |b| < 0, \qquad |c| = 1, \qquad (2)$$

$$|a| + b < 0, \qquad |c| < 1, \qquad \tau < \tau_0, \qquad (3)$$

where $\tau_0 = \arccos(\frac{a-bc}{ac-b})/(\frac{b^2-a^2}{1-c^2})^{\frac{1}{2}}$.

The numerical stability by θ-method of Equation (1.1) was firstly studied by Huan Su et al (2013) [8]. Based on the conditions given in [9] ($a \leq b < -a$ and $|c| < 1$), it is shown in [8] that Equation (1.1) is numerically asymptotically stable for any $\tau > 0$, $\theta \in [\frac{1}{2}, 1]$ and $m = \frac{\tau}{h}$ or for any $\tau < \tau_*$, $\theta \in [0, \frac{1}{2}[$ and m positive integer, where h is the stepsize of discretization and $\tau_* = \min(\frac{2m(1-c)}{(a+b)(2\theta-1)}, \frac{2m(1+c)}{(a-b)(2\theta-1)})$, Theorem (4) in [8].

By Euler's method (corresponding to θ = 0), Jana Hrabalova (2013), [10] studied a numerical stability region in the case $a = 0$. Using Proposition (2.1) given by Jan Cermak et al [11], it is shown that Equation (1.1) is asymptotically stable if and only if,

$$b < 0, \qquad |bh - 2c| < 2, \qquad \tau < \overline{\tau}_0, \qquad (1.2)$$

with $\overline{\tau}_0 = \frac{h \arccos \frac{2c-bh}{2}}{\arccos(\frac{2(1-c^2)+2bch-b^2h^2}{2(1-c^2+bch)})}$.

In (2014), Jan Cermak and Jana Hrabalova [7] have extended the paper [10] to the case $a \neq 0$ and $\theta = \frac{1}{2}$.

The object in this paper is to extend the numerical results of [7]. By θ - method discretization for $\theta \in [0, 1]$, we derive a necessary and sufficient optimal conditions on the parameters a, b, c and τ, in order to preserve the asymptotic stability conditions (1), (2) and (3) of Equation (1.1).

The paper is organized as follows. In Section 2 we provide the θ-method discretization of (1.1) and derive a necessary and sufficient conditions for its asymptotic stability. Discussion and conclusions are given in Section 3. Lastly, we give a numerical examples in Section 4.

2 THE θ -METHOD DISCRETIZATION

In this section, we discretize Equation (1.1) by θ-method. Let us consider a mesh $t_n = nh$, $n = 0, 1, 2, ...$, where $h > 0$ is a stepsize of the method and let $m \geq 1$ an integer. The parameters τ, m and h are related by $m = \frac{\tau}{h}$. The θ-method discretization for a delay differential equation

$$x'(t) = f(t, x(t), x(t-\tau)) \qquad (2.1)$$

is a formula of the form

$$x_{n+1} - x_n = h\theta f(t_{n+1}, x_{n+1}, x_{n-m+1}) + h(1-\theta)f(t_n, x_n, x_{n-m}). \tag{2.2}$$

The application of the θ-method discretizotion to (1.1) is the corresponding formula,

$$x_{n+1} - x_n - c(x_{n-m+1} - x_{n-m}) = \frac{\tau\theta}{m}(ax_{n+1} + bx_{n-m+1}) + \frac{\tau(\theta-1)}{m}(ax_n - bx_{n-m}), \tag{2.3}$$

where $x_n = x(t_n)$ is an approximation of the solution x of Equation (1.1).
Then Equation (2.3) becomes

$$x_{n+1} + \alpha x_n + \beta x_{n-m+1} + \gamma x_{n-m} = 0 \tag{2.4}$$

where

$\alpha = -\frac{m+a\tau(1-\theta)}{m-a\tau\theta}, \qquad \beta = -\frac{mc+b\tau\theta}{m-a\tau\theta}, \qquad \gamma = -\frac{b\tau(1-\theta)-mc}{m-a\tau\theta}$ (*)

Recall that Equation (2.4) is asymptotically stable if $\lim_{n\to\infty} x_n = 0$ for any solution x_n of (2.4). It is well known that Equation (2.4) is asymptotically stable if and only if all the roots of its characteristic polynomial

$$\lambda^{m+1} + \alpha\lambda^m + \beta\lambda + \gamma = 0, \tag{2.5}$$

are located inside the open unit - disk [12].

The following proposition gives necessary and sufficient conditions for all the roots of (2.5) lie within the open unit - disk.

2.1 Proposition [11]

Let α, β and γ be real constants and m be a positive integer. Then all the roots of (2.5), lie inside the unit disk if and only if one of the following conditions holds:

$(C_1)\begin{cases} 1+\alpha+\beta+\gamma > 0, & 1+\alpha-\beta-\gamma > 0, \\ 1-\alpha+\beta-\gamma > 0, & 1-\alpha-\beta+\gamma > 0, \end{cases}$

$(C_2)\begin{cases} 1+\alpha+\beta+\gamma > 0, & 1+\alpha-\beta-\gamma = 0, \\ 1-\alpha+\beta-\gamma > 0, & 1-\alpha-\beta+\gamma > 0, \end{cases}$

$(C_3)\begin{cases} 1+\alpha+\beta+\gamma > 0, & 1+\alpha-\beta-\gamma > 0, \\ 1-\alpha+\beta-\gamma = 0, & 1-\alpha-\beta+\gamma > 0 \end{cases}$ and m is any positive odd integer,

$(C_4)\begin{cases} 1+\alpha+\beta+\gamma > 0, & 1+\alpha-\beta-\gamma > 0, \\ 1-\alpha+\beta-\gamma > 0, & 1-\alpha-\beta+\gamma = 0 \end{cases}$ and m is any positive even integer,

$(C_5)\begin{cases} 1+\alpha+\beta+\gamma > 0, & 1+\alpha-\beta-\gamma < 0, \\ 1-\alpha+\beta-\gamma > 0, & 1-\alpha-\beta+\gamma > 0 \end{cases}$ and m is any positive integer such that,

$$m < \arccos\frac{\alpha^2-\beta^2+\gamma^2-1}{2|\alpha\gamma-\beta|} / \arccos\frac{\alpha^2-\beta^2-\gamma^2+1}{2|\alpha-\gamma\beta|}, \tag{2.6}$$

$(C_6)\begin{cases} 1+\alpha+\beta+\gamma > 0, & 1+\alpha-\beta-\gamma > 0, \\ 1-\alpha+\beta-\gamma < 0, & 1-\alpha-\beta+\gamma > 0 \end{cases}$ and m is any positive odd integer such that (2.6) holds ,

$(C_7)\left\{\begin{array}{ll} 1+\alpha+\beta+\gamma>0, & 1+\alpha-\beta-\gamma>0, \\ 1-\alpha+\beta-\gamma>0, & 1-\alpha-\beta+\gamma<0 \end{array}\right.$ and m is any positive even integer such that (2.6) holds.

Denote by

$$\overline{\tau}(m)=\frac{tan^k \frac{1}{2m}\arccos[\frac{2m(a-bc)+\tau(a^2+b^2)(1-2\theta)}{2|m(b-ac)+ab\tau(1-2\theta)|}]}{[\frac{b^2-a^2}{4m^2(1-c^2)+4m\tau(a+bc)(1-2\theta)+\tau^2(a^2-b^2)(1-2\theta)^2}]^{\frac{1}{2}}}, \qquad with \qquad k=sign(1-|c|), \tag{2.7}$$

$\tau_1=\tau_1(m)=\frac{2m(1-c)}{(a+b)(2\theta-1)}$, $\qquad \tau_2=\tau_2(m)=\frac{2m(1+c)}{(a-b)(2\theta-1)}$,
$\tau_*=\tau_*(m)=\min(\tau_1,\tau_2)$, $\qquad \tau^*=\tau^*(m)=\max(\tau_1,\tau_2)$
$\overline{\tau_1}=\overline{\tau_1}(m)=\frac{2m(1+(-1)^m c)}{(a+(-1)^{m+1}b)(2\theta-1)}$, $\qquad \overline{\tau_2}=\overline{\tau_2}(m)=\frac{2m(1+(-1)^{m+1}c)}{(a+(-1)^m b)(2\theta-1)}$.

For m sufficiently large, we have $m-a\theta\tau>0$.

By replacing α, β and γ by their respective values, conditions (C_1) to (C_7) give us:

2.2 Theorem

Equation (2.4) is asymptotically stable if and only if one of the following conditions hold:

$$(C'_1)\left\{\begin{array}{l} a\le b<-a, \\ 2m(1-c)+(a+b)(1-2\theta)\tau>0, \\ 2m(1+c)+(a-b)(1-2\theta)\tau>0, \end{array}\right.$$

$(C'_2)\left\{\begin{array}{l} a<b<-a, \\ 2m(1-c)+(a+b)(1-2\theta)\tau=0, \\ 2m(1+c)+(a-b)(1-2\theta)\tau>0 \end{array}\right.$ and m is any positive odd integer,

$(C'_3)\left\{\begin{array}{l} a<b<-a, \\ 2m(1-c)+(a+b)(1-2\theta)\tau>0, \\ 2m(1+c)+(a-b)(1-2\theta)\tau=0 \end{array}\right.$ and m is any positive even integer,

$(C'_4)\left\{\begin{array}{l} b<a<-b, \\ 2m(1-c)+(a+b)(1-2\theta)\tau>0, \\ 2m(1+c)+(a-b)(1-2\theta)\tau>0 \end{array}\right.$ and $\tau<\overline{\tau}(m)$,

$(C'_5)\left\{\begin{array}{l} a<b<-a, \\ (-1)^m[2m(1-c)+(a+b)(1-2\theta)\tau]>0, \\ (-1)^{m+1}[2m(1+c)+(a-b)(1-2\theta)\tau]>0 \end{array}\right.$ and $\tau<\overline{\tau}(m)$.

Proof. By Proposition [11], we have to analyse conditions (C_1) -(C_7). By simple calculation we found.

$1+\alpha+\beta+\gamma=\frac{-(a+b)\tau}{m-a\tau\theta}$, $\qquad 1+\alpha-\beta-\gamma=\frac{-(a-b)\tau}{m-a\tau\theta}$, $\qquad 1-\alpha+\beta-\gamma=\frac{2m(1-c)+(a+b)\tau(1-2\theta)}{m-a\tau\theta}$ and $1-\alpha-\beta+\gamma=\frac{2m(1+c)+(a-b)\tau(1-2\theta)}{m-a\tau\theta}$.

So, the conditions (C_1), (C_2) become

$$a\le b<-a,$$

$$2m(1-c)+(a+b)(1-2\theta)\tau>0,$$

$$2m(1+c)+(a-b)(1-2\theta)\tau>0.$$

This justifies (C'_1). Similarly conditions (C_3), (C_4) and (C_5) give respectively (C'_2), (C'_3) and (C'_4) and conditions (C_6) and (C_7) give (C'_5).

The application of the previous Theorem gives us the following results:

2.3 Corollary 1

For $\theta \in [0, \frac{1}{2}[$, Equation (2.4) is asymptotically stable if and only if one of the following conditions holds:

$$a \le b < -a, \quad |c|<1, \quad \tau<\tau_*, \tag{2.8}$$

$$a < b < -a, \quad (-1)^{m+1}c=1, \quad \tau<\overline{\tau_2}, \quad \tau<\overline{\tau}(m) \tag{2.9}$$

$$a < b < -a, \quad |c|<1, \quad (-1)^m(ac+b)>0, \quad \tau=\overline{\tau_1}, \tag{2.10}$$

$$a < b < -a, \quad |c|<1, \quad (-1)^m(ac+b)>0, \quad \overline{\tau}_1<\tau<\overline{\tau}_2, \quad \tau<\overline{\tau}(m), \tag{2.11}$$

$$a < b < -a, \quad (-1)^{m+1}c>1, \quad \tau<\overline{\tau}_2, \quad \tau>\overline{\tau}(m), \tag{2.12}$$

$$b < a < -b, \quad |c|<1, \quad \tau<\tau_1, \quad \tau<\overline{\tau}(m), \tag{2.13}$$

$$b < a < -b, \quad c\le -1, \quad b+ac<0, \quad \tau_2<\tau<\tau_1, \quad \tau>\overline{\tau}(m). \tag{2.14}$$

Proof. For $\theta \in [0, \frac{1}{2}[$, conditions (C'_1) give $a \le b < -a, \quad |c|<1$ and $\tau<\tau_*$. Similarly, conditions (C'_2) and (C'_3) become respectively ($a<b<-a, \quad \tau=\frac{2m(1-c)}{(a+b)(2\theta-1)}, \quad \tau<\frac{2m(1+c)}{(a-b)(2\theta-1)}$ for m is odd integer) and ($a<b<-a, \quad \tau=\frac{2m(1+c)}{(a-b)(2\theta-1)}, \quad \tau<\frac{2m(1-c)}{(a+b)(2\theta-1)}$ for m is even integer), this justifies (2.10). Conditions (C'_4) and (C'_5) already contain the restriction on m. Conditions (C'_4) give ($b<a<-b, \quad \tau<\frac{2m(1-c)}{(a+b)(2\theta-1)}, \quad \tau>\frac{2m(1+c)}{(a-b)(2\theta-1)}$, and m is such that (2.6) holds.

Now we discuss the form of (2.6). Using (*) we can rewrite it as

$$m\arccos\left(\frac{2m^2(1-c^2)+\tau^2(a^2-b^2)(2\theta^2-2\theta+1)+2m\tau(a+bc)(1-2\theta)}{2|m^2(1-c^2)+\tau^2(b^2-a^2)\theta(1-\theta)+m\tau(a+bc)(1-2\theta)|}\right)$$

$$<\arccos\left(\frac{2m(a-bc)+\tau(a^2+b^2)(1-2\theta)}{2|m(b-ac)+ab\tau(1-2\theta)|}\right) \qquad (**)$$

The left-hand side of (**) can be treated by use of the relation,

$$\arccos x = 2\arctan\frac{(1-x^2)^{\frac{1}{2}}}{1+x} \qquad -1\le x\le 1,$$

which results either in

$$2m\arctan\left[\tau\left(\frac{b^2-a^2}{4m^2(1-c^2)+4m\tau(a+bc)(1-2\theta)+\tau^2(a^2-b^2)(2\theta-1)}\right)^{\frac{1}{2}}\right]$$

if

$$m^2(1-c^2)+\tau^2(b^2-a^2)\theta(1-\theta)+m\tau(a+bc)(1-2\theta)>0$$

or in

$$2m \quad arccot\left[\tau\left(\frac{b^2-a^2}{4m^2(1-c^2)+4m\tau(a+bc)(1-2\theta)+\tau^2(a^2-b^2)(2\theta-1)}\right)^{\frac{1}{2}}\right]$$

if

$$m^2(1-c^2)+\tau^2(b^2-a^2)\theta(1-\theta)+m\tau(a+bc)(1-2\theta)<0.$$

Obviously, (2.6) becomes $\tau < \overline{\tau}(m)$ if the first condition holds, or $\tau > \overline{\tau}(m)$ if the second holds.

Then conditions (C'_4) yields (2.13) and (2.14). Similarly, conditions (C'_5) yield (2.9), (2.11) and (2.12).

Remark 1. Let $\theta \in [0.\frac{1}{2}[$. Comparing Corollary 1 and Theorem 2.1 in [7], we see that the θ-method could retain the asymptotical stability for sufficiently large m in the case of conditions (2.8) or (2.13). The conditions ((2.9)-(2.12)) depend on the parity of m and in (2.14) the condition $c \leq -1$ is not necessary. We note that (2.8) is same as the sufficient condition of Theorem 4 in Huan Su et al [8], for $\theta \in [0, \frac{1}{2}[$ and (2.13) is analogous to the formula (20) in Theorem 3.2 in J. Cermak [7] if $\theta = \frac{1}{2}$ and $\overline{\tau}(m)$ converges to τ_0.

2.4 Corollary 2

For $\theta \in]\frac{1}{2}, 1]$, Equation (2.4) is asymptotically stable if and only if one of the following conditions holds:

$$a \leq b < -a, \quad |c| < 1, \tag{2.15}$$

$$a < b < -a, \quad |c| = 1, \tag{2.16}$$

$$a \leq b < -a, \quad |c| \geq 1, \quad \tau > \tau^*, \tag{2.17}$$

$$a < b < -a, \quad (-1)^{m+1}c \geq 1, \quad \tau = \overline{\tau}_1, \tag{2.18}$$

$$a < b < -a, \quad (-1)^{m+1}c > 1, \quad \tau < \overline{\tau}_1, \quad \tau > \overline{\tau}(m), \tag{2.19}$$

$$b < a < -b, \quad |c| < 1, \quad \tau < \tau_2, \quad \tau < \overline{\tau}(m), \tag{2.20}$$

$$b < a < -b, \quad c \geq 1, \quad ac + b < 0, \quad \tau_1 < \tau < \tau_2, \quad \tau > \overline{\tau}(m). \tag{2.21}$$

Proof. For $\theta \in]\frac{1}{2}, 1]$, conditions (C'_1) give ($a \leq b < -a, \quad \tau > \frac{2m(1-c)}{(a+b)(2\theta-1)}$, and $\tau > \frac{2m(1+c)}{(a-b)(2\theta-1)}$), which can be written jointly as (2.15), (2.16) and (2.17). Similarly, conditions (C'_2) and (C'_3) become respectively ($a < b < -a, \quad \tau = \frac{2m(1-c)}{(a+b)(2\theta-1)} \quad \tau > \frac{2m(1+c)}{(a-b)(2\theta-1)}$ in the case m is odd integer) and ($a < b < -a, \quad \tau = \frac{2m(1+c)}{(a-b)(2\theta-1)} \quad \tau > \frac{2m(1-c)}{(a+b)(2\theta-1)}$ in the case m is even integer). Then this check (2.18). Conditions (C'_4) give ($b < a < -b, \quad \tau > \frac{2m(1-c)}{(a+b)(2\theta-1)}, \quad \tau < \frac{2m(1+c)}{(a-b)(2\theta-1)}$ and m is such that (2.6) holds. This justify (2.20) and (2.21). (2.19) is analogous to (C'_5).

Remark 2. Let $\theta \in]\frac{1}{2}, 1]$. By making a comparison between Corollary 2 and Theorem 2.1 in [7] and from a same arguments as for Remark 1, we see that the θ-method could retain the asymptotical stability for sufficiently large m only in the case of conditions (2.15) or (2.16) or (2.20) if $\overline{\tau}(m)$ converges to τ_0 in decreasing.

2.5 Corollary 3

For $\theta = \frac{1}{2}$, Equation (2.4) is asymptotically stable if and only if one of the following conditions holds:

$$a \leq b < -a, \quad |c| < 1, \tag{2.22}$$

$$a < b < -a, \quad (-1)^{m+1}c = 1, \tag{2.23}$$

$$b < a < -b, \quad |c| < 1, \quad \tau < \overline{\tau}(m), \tag{2.24}$$

$$b < a < -b, \quad (-1)^{m+1}c > 1, \quad \tau > \overline{\tau}(m), \tag{2.25}$$

Proof. By replacing θ by $\frac{1}{2}$ in the previous Theorem, conditions (C'_1), (C'_4) and (C'_5) become respectively (2.22), (2.24) and (2.25). Similarly, conditions (C'_2) and (C'_3) yield (2.23).

Note that Corollary 3 is analogous to Theorem 3.2 in [7].

Denote by $\Sigma_\tau(m)$ the region of numerical stability by θ-method and by Σ^*_τ the region of the theoretical stability,

$$\Sigma_\tau(m) = \{(a, b, c) \ \ satisfying \ \ the \ \ conditions\ of\ corollary \ \ 1\ or\ 2\ or\ 3\},$$
$$\Sigma^*_\tau = \{(a, b, c) \ \ satisfying \ \ the \ \ conditions\ of\ Theorem \ \ 2.2\}.$$

2.6 Corollary 4

Let $\theta \in [\frac{1}{2}, 1]$ and $m_1 < m_2$ be arbitrary positive integers. Then

$$\Sigma_\tau(m_2) \subset \Sigma_\tau(m_1) \text{ and } \cap_{m\in\mathbb{N}^*}\Sigma_\tau(m) \supset \Sigma^*_\tau$$

Proof. If $\theta = \frac{1}{2}$, the proof is already done (J. Cermak et al [7]). Let $\theta \in]\frac{1}{2}, 1]$ and $m_1 < m_2$ two arbitrary positive integers and let's show that $\Sigma_\tau(m_2) \subset \Sigma_\tau(m_1)$.

Since the condition (2.15) and (2.16) in Corollary 2 are independent of m, it is enough to consider the delay-dependent part of $\Sigma_\tau(m)$ in (2.20) and to show that $\overline{\tau}(m)$ converges to τ_0 in decreasing (also J. Cermak et al [7]).

Due to the fact that: sign $(m(b - ac) + ab\tau(1 - 2\theta))$ = sign $(m(b - ac)$, for m sufficiently large and as $b - ac < 0$ from the condition (2.20); (2.7) becames

$$\overline{\tau}(m) = \frac{tan\frac{1}{2m}\arccos[\frac{2m(a-bc)+\tau(a^2+b^2)(1-2\theta)}{2m(ac-b)-2ab\tau(1-2\theta)}]}{[\frac{b^2-a^2}{4m^2(1-c^2)+4m\tau(a+bc)(1-2\theta)+\tau^2(a^2-b^2)(1-2\theta)^2}]^{\frac{1}{2}}},$$

of the form:

$$f_\theta(x) = \frac{tan\frac{1}{2x}\arccos[\frac{Ax+B}{Cx+D}]}{[\frac{K}{Ex^2+Gx+H}]^{\frac{1}{2}}}$$

with:

$A = (a - bc),\quad B = \tau(a^2 + b^2)(1 - 2\theta),\quad C = 2(ac - b),\quad D = -2ab\tau(1 - 2\theta),$
$E = 4(1 - c^2)\quad G = 4\tau(a + bc)(1 - 2\theta),\quad H = \tau^2(a^2 - b^2)(1 - 2\theta),\quad K = b^2 - a^2$

Note that, for $\theta = \frac{1}{2}$, $f_{\frac{1}{2}}(x) = 2x\tan\frac{r}{2x}$, with $r \in]0, \pi[$. It's not difficult to see that $f_{\frac{1}{2}}$ is decreasing for $x \in]2, +\infty[$ (J. Cermak et al [7]). We cleam that f_θ is as $f_{\frac{1}{2}}$, decreasing for x sufficiently large.

In fact,

$$f'_\theta(x) = [-\frac{1}{2x^2}\arccos(\frac{Ax+B}{Cx+D})]\times[1+\tan^2(\frac{1}{2x}\arccos(\frac{Ax+B}{Cx+D}))]\times(\frac{K}{Ex^2+Gx+H})^{\frac{1}{2}}/(\frac{K}{Ex^2+Gx+H})$$
$$-[(\frac{K}{Ex^2+Gx+H})^{\frac{1}{2}}]'\times[\tan[\frac{1}{2x}\arccos(\frac{Ax+B}{Cx+D})]/(\frac{K}{Ex^2+Gx+H})$$
$$+\frac{1}{2x}[\arccos(\frac{Ax+B}{Cx+D})]'\times[1+\tan^2(\frac{1}{2x}\arccos(\frac{Ax+B}{Cx+D}))]\times(\frac{K}{Ex^2+Gx+H})^{\frac{1}{2}}/(\frac{K}{Ex^2+Gx+H})$$

It's clear that the sign of the first expression of $f'_\theta(x)$ (lines 1,2) is same as sign $(f'_{\frac{1}{2}}(x))$ negative and sign of the second one (line 3) is same as sign $([\arccos(\frac{Ax+B}{Cx+D})]')$.

By simple calculation we found, sign $([\arccos(\frac{Ax+B}{Cx+D})]')$ = $sign(BC - AD) < 0$.

Consequently, f_θ is decreasing

3 DISCUSSION AND CONCLUSIONS

The stability of many numerical methods for linear NDDEs and DDEs has been extensively studied [13] [14] [15] [16] [17] [18] [19] [20] [21] and the references therein. Recently the problem of necessary and sufficient numerical stability conditions by θ-method discretization for the NDDEs (1.1) is considered in the following situations: $\theta = 0$ and $a = 0$ by J. Hrabalova [10] and $\theta = \frac{1}{2}$ by Jan Cermak et al [7].

In the cas $\theta \in [0, 1]$, Huan Su et al [8] considered only a necessary condition. If the following theoretical asymptotic stability conditions ($a \leq b < -a$ and $|c| < 1$) hold; then, one has also numerical stability [8].

In this paper we can consider that the conditions on the parameters obtained in corollaries 1-3 and the ones obtained by H. I. Freedman et al in [22], permitted a good comparison between the numerical θ-methods stability for $\theta \in [0, 1]$ and the theoretical stability, for the NDDEs (1.1).

Note that Theorem 7.3 in [23] shows that the θ-methods are NGP-stable if and only if $\theta \in [\frac{1}{2}, 1]$. Here, with the help of Corollary 4, we can confirm that in the case $\theta \in [\frac{1}{2}, 1]$, the conditions of corollaries 2 and 3 are optimal to preserve the theoretical asymptotic stability.

4 SOME NUMERICAL SIMULATIONS

In this section, some numerical examples are provided to support our main results in corollaries 1-3. We take $a = -2, b = -1$ as an example and the two cases: $|c| < 1$, $\theta \in [0, 1]$ and $|c| = 1$, $\theta \in]\frac{1}{2}, 1]$.

For the former case, for $|c| < 1$, $\theta \in [0, 1]$ condition (2.8), (2.15) and (2.22) are satisfied. Therefore, by Corollary 1, the origin is asymptotically stable if $\theta \in [0, \frac{1}{2}[$ any $\tau < \tau_*$. And by corollaries 2, 3, the origin is asymptotically stable for $\theta \in [\frac{1}{2}, 1]$ and for any $\tau > 0$. This is shown respectively in Fig. 1 (for $c = 0.5$, $\theta = 0$) and Fig. 2 (for $c = 0.5$, $\theta = \frac{1}{2}$, and $c = 0.5$, $\theta = 1$).

For the second case, for $c = 1$ or $c = -1$, conditions (2.16) are satisfied, and by Corollary 2 the origin is asymptotically stable for any $\tau > 0$, and $\theta \in]\frac{1}{2}, 1]$. This is shown in Fig. 3 (respectively Fig.4), for $\theta = 1$ and $\tau = 1$ or $\tau = 5$ (respectively for $\theta = 1$, or $\theta = \frac{3}{4}$ and $\tau = 1$).

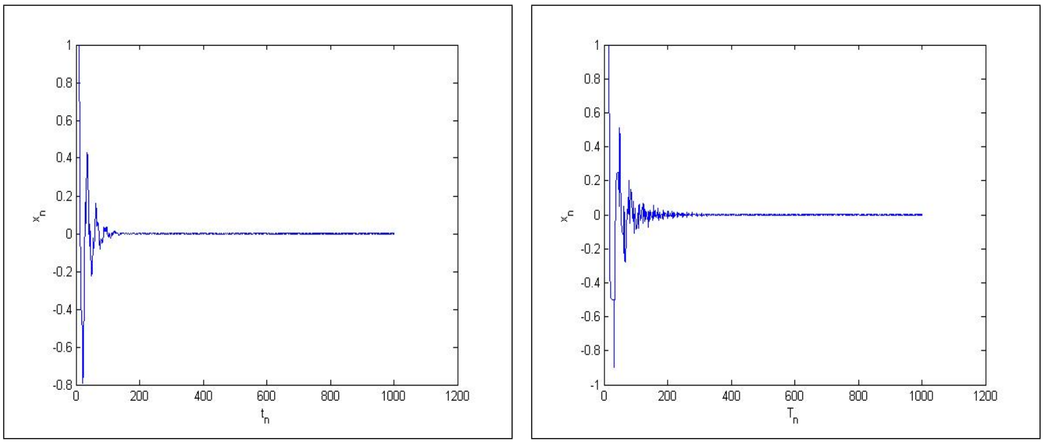

Fig. 1. **Stable solutions of (2.4) for $\tau = 2$, $m = 10$ and $\tau = 4$, $m = 15$.**

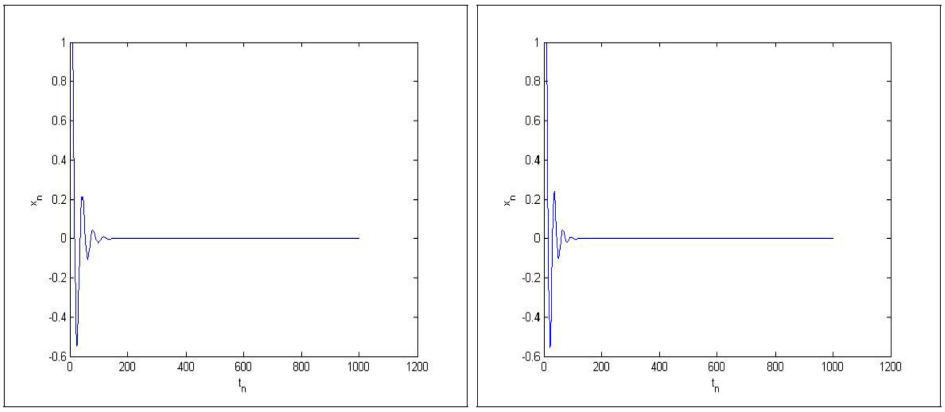

Fig. 2. **Stable solutions of (2.4) for** $\theta = 0.5$, $\tau = 1$, $m = 10$ **and** $\theta = 1$, $\tau = 2$, $m = 10$, **when** $c = 0.5$.

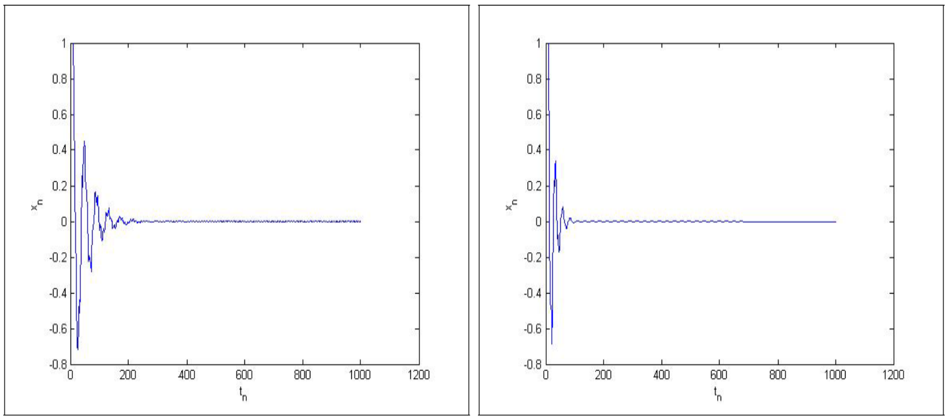

Fig. 3. **Stable solutions of (2.4) for** $\theta = 1$, $\tau = 1$, **and** $\tau = 5$, **when** $c = 1$.

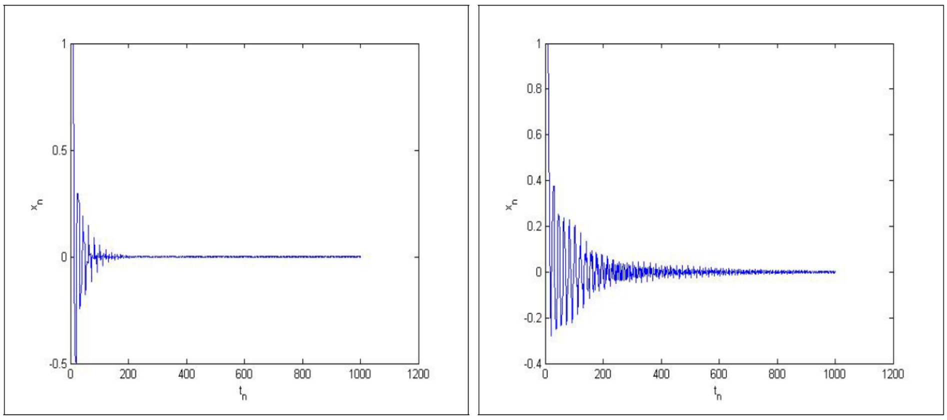

Fig. 4. **Stable solutions of (2.4) for** $\theta = 1$, $\tau = 5$ **and** $\theta = \frac{3}{4}$, $\tau = 1$, **when** $c = -1$.

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COMPETING INTERESTS

The authors declare that no competing interests exist.

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