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Extremal total eccentricity of k-apex trees

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Abstract: In a simple connected graph G, eccentricity of a vertex is one of the first, distance-based invariants. The eccentricity of a vertex v in a connected graph G is the maximum distance of the vertex v to any other vertex u. The total eccentricity of the graph G is the sum of the all vertex eccentricities. A graph G is called an apex tree if it has a vertex x such that $G - x$ is a tree. In this work we have found the graph having extremal total eccentricity of k-apex trees.

Keywords: k-apex trees, total eccentricity, extreaml graphs, eccentricity.

MSC: 26B25, 26A33, 26A51, 33E12.

1. Introduction

We consider only simple and connected graphs. In a graph G, the eccentricity [1–3] of a vertex v is one of the first distance-based, invariants. The *eccentricity* of a vertex v in a connected graph G is the distance function as

$$ecc_G(v) = \max_{u \in V(G)} d(u, v).$$

Total eccentricity of a graph G is the sum of all vertex eccentricities

$$Ecc(G) = \sum_{z \in V(G)} ecc_G(z).$$

Fathalikhani *et al.* [4] computed total eccentricity of some graph operations, and found bound for that of tensor product. Eccentricity, center and radius computations on the cover graphs of distributive lattices were presented by Suzuki and McDermind [5]. Extremal trees, unicyclic, bicyclic graphs and extremal conjugated trees with respect to total eccentricity index were presented by Farooq *et al.* in [6].

If a graph G contains a vertex x, such that $G - x$ is a tree, then it is called *apex tree*. The vertex x, is called an *apex vertex* of G. A tree is always an apex tree, therefore a *non-trivial apex tree* is an apex tree which itself is not a tree. For any natural number $k \geq 1$, a graph G is called a k-apex tree if there exists a set X of k elements, and subset of $V(G)$ such that $G - X$ is a tree, while for any $Y \subseteq V(G)$ with $|Y| < k$, $G - Y$ is not a tree. Any vertex from the set X is called a k-apex vertex. $T_k(n)$ denotes the set of all k-apex trees of order n, for $k \geq 1$ and $n \geq 3$. Sharp upper bound for the Randić (connectivity) index of k-apex trees for $k \geq 2$ were discovered by Akhter *et al.* [7]. Extremal first reformulated Zagreb index of k-apex trees was discovered by Akhter *et al.* in [8]. In this paper we have found minimal total eccentricity of k-apex trees and corresponding graphs. The join of two vertex-disjoint graphs G and H is the graph $G + H$ with $V(G + H) = V(G) \cup V(H)$ and the edges of $G + H$ are all edges of graphs G and H and the edges obtained by joining each vertex of G with each vertex of H. Let $G \in T_k(n)$ and X be the set of k-apex vertices, by the graph X_G we mean the subgraph of G whose vertex set is X. The complete graph with vertex set X will be denoted by KX_G. The tree obtained from G by deleting apex vertices will be denoted by T_G. We shall denote the star graph whose vertex set is $V(G) - X$ by S_G. If $G \in T_k(n)$, then $X_G + T_G$ and $KX_G + S_G$ are also k-apex trees. A tree of n vertices will be denoted by T_n. We shall denote the eccentricity of a vertex v in a graph G by $ecc_G(v)$.

2. Minimal total eccentricity of k-apex trees

The following is a fundamental lemma, and is used to obtain many useful results.

Lemma 1. If $u, v \in V(G)$ are not adjacent, then $Ecc(G + uv) \leq Ecc(G)$.

The proof of above lemma is obvious.

Lemma 2. If $G \in T_k(n)$, $k \geq 1$ and $n - k \geq 3$ then $Ecc(G) \geq Ecc(KX_G + S_G)$.

Proof. As $V(G) = V(X_G + T_G)$ and G is a subgraph of $X_G + T_G$ so by Lemma 1

$$Ecc(G) \geq Ecc(X_G + T_G). \tag{1}$$

As graph $KX_G + T_G$ is either same as $X_G + T_G$ or it is obtained by adding edges in $X_G + T_G$ therefore by Lemma 1

$$Ecc(X_G + T_G) \geq Ecc(KX_G + T_G). \tag{2}$$

If T_G is not a star then for every vertex $v \in V(G) - X$, $ecc_{KX_G + T_G}(v) \geq 2$ and if $T_G = S_G$, then $ecc_{KX_G + S_G}(v) \leq 2$, therefore

$$Ecc(KX_G + T_G) \geq Ecc(KX_G + S_G). \tag{3}$$

Combining inequalities 1, 2 and 3 we have the result.

$$Ecc(G) \geq Ecc(KX_G + S_G).$$

□

The following lemma gives a relation between eccentricities of join of graphs and eccentricities of graphs.

Lemma 3. If K_n and S_m are vertex disjoint graphs for $n \geq 2$ and $m \geq 2$, then $Ecc(K_n + S_m) = Ecc(K_n) + Ecc(S_m)$.

Proof. For any vertex $v \in V(K_n)$, we have $ecc_{K_n + S_m}(v) = ecc_{K_n}(v) = 1$ and for any vertex $u \in V(S_m)$, we have $ecc_{K_n + S_m}(u) = ecc(S_m)$. Thus

$$Ecc(K_n + S_m) = \sum_{v \in V(K_n + S_m)} ecc_{K_n + S_m}(v).$$

As graphs K_n and S_m are vertex disjoint, therefore

$$Ecc(K_n + S_m) = \sum_{v \in V(K_n)} ecc_{K_n + S_m}(v) + \sum_{v \in V(S_m} ecc_{K_n + S_m}(v).$$

Since for any vertex $v \in K_n$, $ecc_{K_n + Sm}(v) = ecc_{K_n}(v)$ and for any vertex $u \in (S_m)$, $ecc_{K_n + S_m}(u) = ecc_{S_m}(u)$ therefore

$$Ecc(K_n + S_m) = \sum_{v \in V(K_n)} ecc_{K_n}(v) + \sum_{v \in V(S_m)} ecc_{S_m}(v)$$

and hence

$$Ecc(K_n + S_m) = Ecc(K_n) + Ecc(S_m).$$

□

Theorem 1. *If $G \in T_k(n)$, $k \geq 1$ and $n - k \geq 3$, then*

$$Ecc(G) \geq 2n - k - 1$$

and equality holds if there are $k + 1$ vertices of eccentricity 1 and all other vertices are of eccentricity 2.

Proof. For any graph $G \in T_k(n)$ and for given conditions on k and n, by Lemma 2, we have

$$Ecc(G) \geq Ecc(KX_G + S_G).$$

As for each $v \in X$, $ecc_{KX_G+S_G}(v) = 1$, for one vertex in $V(G) - X$, eccentricity is one and for all other $n - k - 1$ vertices in $V(G) - X$, eccentricity is 2, therefore

$$Ecc(G) \geq 2n - k - 1.$$

□

Theorem 2. *If* $G \in T_k(n)$, $k \geq 1$ *and* $n - k = 2$, *then*

$$Ecc(G) \geq n$$

and equality holds if all vertices are of eccentricity 1.

Proof. By Lemma 2, for any k-apex tree G, we have $Ecc(G) \geq Ecc(KX_G + S_G)$. In this case star will be of order 2 and therefore for every vertex $v \in KX_G + S_G$, $ecc(v) = 1$ and hence

$$Ecc(G) \geq n.$$

□

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