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# GLOBAL SOLUTION AND ASYMPTOTIC BEHAVIOUR FOR A WAVE EQUATION TYPE $p$-LAPLACIAN WITH MEMORY

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Abstract. In this work we study the global solution, uniqueness and asymptotic behaviour of the nonlinear equation

$$u_{tt} - \Delta_p u = \Delta u - g * \Delta u$$

where $\Delta_p u$ is the nonlinear p-Laplacian operator, $p \geq 2$ and $g * \Delta u$ is a memory damping. The global solution is constructed by means of the Faedo-Galerkin approximations taking into account that the initial data is in appropriated set of stability created from the Nehari manifold and the asymptotic behavior is obtained by using a result of P. Martinez based on new inequality that generalizes the results of Haraux and Nakao.

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*Key words and phrases:* $p$-Laplacian operator; global solution; asymptotic behaviour; memory.

## 1. Introduction

Throughout this paper we omit the space variable $x$ of $u(x,t)$, simply denote $u(x,t)$ by $u(t)$ when no confusion arises and $c$ denotes various positive constants depending on the known constants and may be different at each appearance. We use the Sobolev space with its properties as in R. A. Adams [1] and H. Brezis [2]. Let $\Omega \in \mathbb{R}$ be a open and bounded interval, $2 \leq p < \infty$ and $p'$

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such that $\dfrac{1}{p}+\dfrac{1}{p'}=1$. The duality pairing between the space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$ will be denoted using the form $\langle\,\cdot\,,\,\cdot\,\rangle_p$. According to Poincarè's inequality, the standard norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ is equivalent to the norm $\|\nabla\cdot\|_p$ on $W_0^{1,p}(\Omega)$. Henceforth, we put $\|\cdot\|_{W_0^{1,p}(\Omega)}=\|\nabla\cdot\|_p$. We denote $\|\cdot\|_{L^2(\Omega)}=|\cdot|_2$ and the usual inner product by $(\,\cdot\,,\,\cdot\,)$. We denote the $p$-Laplacian operator by $\Delta_p u$, which can be extended to a monotone, bounded, hemicontinuous and coercive operator between the spaces $W_0^{1,p}(\Omega)$ and its dual by

$$-\Delta_p\colon W_0^{1,p}(\Omega)\to W^{-1,p'}(\Omega)$$

$$\langle-\Delta_p u,v\rangle_p=\int_\Omega|\nabla u|^{p-2}\nabla u\nabla v\,\mathrm{d}x$$

Nonlinear hyperbolic problems involving the $p$-Laplacian are becoming the object of increasing interest only in recent years. The existence of a global solution for wave equation of $p$-Laplacian type

$$u_{tt}-\Delta_p u=0 \tag{1}$$

without an additional dissipation term is an open problem. For $n=1$, M. Derher [3] proved the local in time existence of solution and showed by a generic counter-example that the global in time solution can not be expected.

Adding a strong damping $(-\Delta u_t)$ in (1) the well-posedness and asymptotic behavior was studied by J. M. Greenberg [4]. In fact, the strong damping plays an important role on the existence and stability for $p$-Laplacian wave equation see for instance for $n\geq 2$ [5, 6, 7, 8, 9, 10, 11, 12]. Nevertheless, if the strong damping is replaced by a weaker damping $(u_t)$, then global existence and uniqueness are only know for $n=1,2$, see [13, 14]. For the intermediary damping given by $((-\Delta)^\alpha u_t)$, with $0<\alpha\leq 1$ in [15] was proved the global solution depending on the growth of a forcing term. The background of these problems are in physics, especially in solid mechanics.

From what we know this is the first time that a alternative damping for wave equation with the $p$-Laplacian operator is considered. In this work we consider a memory damping, acting only on $\Delta u$ given by the usual convolution

$$g*\Delta u(x,t)=\int_\Omega g(t-s)\Delta u(x,s)\,\mathrm{d}s$$

with the kernel $g$ as real-valued function.

We have interest in proving the existence of a global solution and energy decay to the problem

$$u_{tt}-\Delta_p u{=}\Delta u-g*\Delta u\quad\text{in}\quad\Omega\times[0,\infty), \tag{2}$$

$$u(x,0){=}u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega, \tag{3}$$

$$u(x,t){=}0\quad\text{on}\quad\partial\Omega\times[0,\infty). \tag{4}$$

This paper is organized as follows. Section 2 deals with the potential well, we introduce the stability set for the problem. In the Section 3 we introduce some notations and preliminaries results. In the section we introduce a suitable Galerkin basis. In the Section 5 we prove the existence of solution by Faedo-Galerkin method and finally in the Section 6 we use the result of P. Martinez [16] that generalizes the results of Haraux [17] and Nakao [18] to prove the energy decay in a appropriate set of stability.

## 2. The Potential Well

It is well known that the energy of a PDE system is, in some sense, split into kinetic and potential energy. Following the idea of Y. Ye [9] we are able to construct a set of stability as follows. We will prove that there is a valley or a well of depth $d$ created in the potential energy. If this height $d$ is strictly positive, we find that, for solutions with initial data in the good part of the well, the potential energy of the solution can never escape the well. In general, it is possible for the energy from the source term to cause the blow-up in finite time. However in the good part of the well it remains bounded. As a result, the total energy of the solution remains finite on any time interval $[0, T)$, which provides the global existence of the solution. We started by introducing the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{2}\int_\Omega \left(g * |\nabla u|^2\right)(t)\,\mathrm{d}x. \tag{5}$$

For $u \in W_0^{1,p}(\Omega)$ we define the functional

$$J(\lambda u) = \frac{\lambda^p}{p}\|\nabla u\|_p^p - \frac{\lambda}{2}\int_\Omega \left(g * |\nabla u|^2\right)(t)\,\mathrm{d}x, \qquad 0 < \lambda \le 1. \tag{6}$$

Associated with the $J$ we have the well known *Nehari Manifold* given by

$$\mathcal{N} \stackrel{\text{def}}{=} \left\{ u \in W_0^{1,p}(\Omega)/\{0\} \;:\; \left[\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u)\right]_{\lambda=1} = 0 \right\}.$$

From (6) we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u) = \lambda^{p-1}\|\nabla u\|_p^p - \frac{1}{2}\int_\Omega \left(g * |\nabla u|^2\right)(t)\,\mathrm{d}x,$$

then

$$\mathcal{N} \stackrel{\text{def}}{=} \left\{ u \in W_0^{1,p}(\Omega)/\{0\} \;:\; \|\nabla u\|_p^p = \frac{1}{2}\int_\Omega \left(g * |\nabla u|^2\right)(t)\,\mathrm{d}x \right\}.$$

We define as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [19],

$$d \stackrel{\text{def}}{=} \inf_{u \in W_0^{1,p}(\Omega)/\{0\}} \sup_{0 \le \lambda} J(\lambda u).$$

It is well-known that the depth of the well $d$ is a strictly positive constant, see [[20], Theorem 4.2], and

$$d = \inf_{u \in \mathcal{N}} J(u).$$

In fact, in our problem, the solution of $\dfrac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) = 0$ is

$$\lambda_* = \left[\frac{\frac{1}{2}\int_\Omega \left(g * |\nabla u|^2\right)(t)\,\mathrm{d}x}{\|\nabla u\|_p^p}\right]^{\frac{1}{p-1}}.$$

We have

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}J(\lambda u) = (p-1)\lambda^{p-2}\|\nabla u\|_p^p > 0,$$

and then $\lambda_*$ is a global minimum.

For $p \geq 2$, $J(\lambda_* u) < 0$, so we introduce the sets

$$\mathcal{W}_1 = \{u \in W_0^{1,p}(\Omega); J(\lambda_* u) \leq J(\lambda u) \leq 0\}$$

and

$$\mathcal{W}_2 = \{u \in W_0^{1,p}(\Omega); 0 < J(\lambda u)\}.$$

The potential well is defined by $\mathcal{W} = \{u \in W_0^{1,p} \ : \ J(u) < d\} \cup \{0\}$ and partition it into two sets

$$V = \left\{u \in \mathcal{W} \ : \ \frac{1}{2}\int_\Omega \left(g * |\nabla u|^2\right)(t)\,\mathrm{d}x \leq \|\nabla u\|_p^p\right\} \cup \{0\},$$

$$W = \left\{u \in \mathcal{W} \ : \ \|\nabla u\|_p^p < \frac{1}{2}\int_\Omega \left(g * |\nabla u|^2\right)(t)\,\mathrm{d}x\right\}.$$

We will refer to $V$ as the "good" part of the well and $W$ as the "bad" part of the well. Then we define by $V$ the set of stability for the problem (2)-(4).

## 3. Preliminaries

We introduce the symbols "$\Box$" and "$\diamond$" which denote the following convolutions respectively

$$(g\Box h)(t) \stackrel{\text{def}}{=} \int_0^t g(t-s)|h(t)-h(s)|^2\,\mathrm{d}s,$$

$$(g \diamond h)(t) \stackrel{\text{def}}{=} \int_0^t g(t-s)\left(h(t)-h(s)\right)\mathrm{d}s.$$

We state two basic results, see [21], that will be used in the sequel.

**Lemma 3.1.** *For any functions $g, h \in C\left([0,\infty], \mathbb{R}\right)$ we have that*

$$(g*h)(t) = \left(\int_0^t g(s)\,\mathrm{d}s\right)h(t) - (g \diamond h)(t)$$

$$|(g \diamond h)(t)|^2 \leq \left(\int_0^t |g(s)|\,\mathrm{d}s\right)(|g|\Box h)(t)$$

**Lemma 3.2.** *For $g, h \in C([0,\infty],\mathbb{R})$ we have*

$$2(g*h)(t)h'(t) = (g'\Box h)(t) - g(t)|h(t)|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\int_0^t g(s)\,\mathrm{d}s\right)|h(t)|^2 - (g\Box h)(t)\right]$$

From now and on, the function $g$ is of exponential type, this is, $g > 0$ and $\exists\ c_i > 0$, $(i = 0, 1)$ such that

$$-c_0 g(t) \leq g'(t) \leq -c_1 g(t) \quad \text{and} \quad 1 - \int_0^\infty g(t)\,\mathrm{d}t < \infty.$$

The energy of the problem (2)-(4) is defined as

$$E(t) \overset{\text{def}}{=} \frac{1}{2}|u'(t)|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla u(t)|_2^2 + \frac{1}{2}(g\Box\nabla u)(t).$$

Now we present the result of P. Martinez [16] on decay rate estimates for dissipative system that will used in the section 6.

**Lemma 3.3.** *Let $E\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function such that*

$$\phi(0) = 0 \text{ and } \phi(t) \to +\infty \text{ as } t \to +\infty.$$

*Assume that there exist $q \geq 0$ and $A > 0$ such that*

$$\int_S^{+\infty} E(t)^{q+1}\phi'(t)\,\mathrm{d}t \leq AE(S),\ \ 0 \leq S < +\infty.$$

*Then we have*

$$E(t) \leq cE(0)\left(1 + \phi(t)\right)^{\frac{-1}{q}},\ \forall\ t \geq 0 \text{ if } q > 0$$

*and*

$$E(t) \leq cE(0)e^{-w\phi(t)},\ \forall\ t > 0 \text{ if } q = 0,$$

*where $c$ and $w$ are positive constants independent of the initial energy $E(0)$.*

The energy of the problem (2)-(4) is defined as

$$E(t) \overset{\text{def}}{=} \frac{1}{2}|u'(t)|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla u(t)|_2^2 + \frac{1}{2}(g\Box\nabla u)(t).$$

## 4. The Galerkin basis

Denote by

$\mathcal{K}_j = \{K \subset \{u \in L^p(\Omega) : ||u||_p = 1\} : K \text{ is compact, symmetric and } \gamma(K) \geq j\}$,

where $\gamma(G) = \inf\{m : \exists\,\phi : G \to \mathbb{R}/\{0\}, \phi \text{ odd continuous function}\}$ denotes the Krasnoselski genus. In [22] it is proved that

$$\lambda_j = \inf_{G\in\mathcal{K}_j} \sup_{u\in G} ||\nabla u||_p^p$$

is a sequence of eigenvalue of the $p$-Laplacian. $-\Delta_p\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a monotone, coercive and hemicontinuous operator on $W_0^{1,p}(\Omega)$. Minty-Browder theorem, see [23], guarantees the existence of a basis $(w_j)_{j=1}^{\infty}$ for $W_0^{1,p}(\Omega)$ given by the solution of the stationary problem

$$\begin{aligned} -\Delta_p w_j &= \lambda_j w_j, \\ w_j(0) &= w_{0j}. \end{aligned}$$

Using

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega) \tag{7}$$

with continuous and dense injection for $1 < p < \infty$, see [2], this basis can be extended on $L^2(\Omega)$ as a basis of Galerkin to Laplacian operator.
In fact, from Sobolev immersion we have

$$W_0^{\nu,q}(\Omega) \hookrightarrow W_0^{\nu-k,q_k}(\Omega),\ \frac{1}{q_k} = \frac{1}{q} - \frac{k}{n}.$$

Choosing $q_k = p$, $\nu - k = 1$ and $q = 2$ we get

$$\nu = 1 + \frac{n}{2} - \frac{n}{p} = 1 + \frac{n(p-2)}{2p} > 0$$

and we obtain a Hilbert Space $H_0^{\nu}(\Omega)$ such that

$$H_0^{\nu}(\Omega) = W_0^{\nu,2}(\Omega) \hookrightarrow W_0^{1,p}(\Omega).$$

Let $s$ an integer for which $s > \nu$. We have

$$H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

By Rellich-Kondrachov theorem, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, so the immersion $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is also. From spectral theory, there exists a operator defined by

$$\{H_0^s(\Omega),\ L^2(\Omega),\ (\!(\cdot,\cdot)\!)_{H_0^s(\Omega)}\}$$

and a sequence of eigenvectors $(v_j)_{j\in\mathbb{N}}$ of this operator, such that

$$(\!(v_j, v)\!)_{H_0^s(\Omega)} = \lambda_j (v_j, v),\ \text{for all } v \in H_0^s(\Omega)$$

with $\lambda_j > 0$, $\lambda_j \leq \lambda_{j+1}$, and $\lambda_j \to +\infty$ as $j \to +\infty$. Moreover $(v_j)_{j\in\mathbb{N}}$ is a complete orthonormal system in $L^2(\Omega)$ and $\left(w_j = \frac{v_j}{\sqrt{\lambda_j}}\right)_{j\in\mathbb{N}}$ is a complete orthonormal system in $H_0^s(\Omega)$. Then $(w_j)_{j\in\mathbb{N}}$ yields a "Galerkin basis" for both $W_0^{1,p}(\Omega)$ and $L^2(\Omega)$.

## 5. Global Solution

### 5.1. Existence.

**Theorem 5.1.** *Given $u_0 \in V$, $u_1 \in L^2(\Omega)$ there exists a function*

$$u \colon \Omega \times (0,T) \to \mathbb{R}$$

*such that*

$$\begin{array}{l} u \in L^\infty(0,T;W_0^{1,p}(\Omega)), \qquad u' \in L^\infty(0,T;L^2(\Omega)), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) \ a.e.\ in\ \Omega, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(u_t,v)+\langle -\Delta_p u, v\rangle_p + (-\Delta u, v) + (g * \Delta u, v) = 0,\ \forall\ v \in W_0^{1,p}(\Omega)\ in\ D'(0,T). \end{array}$$

*Proof.* Now, for each $m \in \mathbb{N}$, let us put $V_m = \operatorname{Span}\{w_1, w_2, \ldots, w_m\}$. We search for a function $u_m(t) = \sum_{j=1}^{m} k_{jm}(t) w_j$ such that for any $v \in V_m$, $u_m(t)$ satisfies the approximate equation

$$(u_m''(t), v) + \langle -\Delta_p u_m(t), v\rangle_p + (-\Delta u_m(t), v) + (g * \Delta u_m(t), v) = 0, \qquad (8)$$

with the initial conditions $u_m(0) = u_{0m}$ and $u_m'(0) = u_{1m}$, where $u_{0m}$ e $u_{1m}$ are closed in $V_m$ so that

$$w_{0m} \to u_0 \ \in \ W_0^{1,p}(\Omega) \ \text{ and } \ u_{1m} \to u_1 \ in\ L^2(\Omega).$$

Putting $v = w_i$, $i = 1, 2, \ldots, m$, and using

$$\begin{aligned} u_m''(t) &= \sum_{j=1}^{m} k_{jm}''(t) w_j(x), \\ \Delta u_m(t) &= \sum_{j=1}^{m} k_{jm}(t) \Delta w_j(x), \\ \Delta_p u_m(t) &= \sum_{j=1}^{m} k_{jm}(t) \Delta_p w_j(x), \\ (g * \Delta u_m)(t) &= \sum_{j=1}^{m} (g * k_{jm})(t) \Delta w_j(x), \end{aligned}$$

we observe that (8) is a system of ODEs in the variable $t$ and has a local solution $u_m(t)$ in a interval $[0, t_m)$, by virtue of Carathéodory's theorem, see [24]. In the next step we obtain priori estimates for the solution $u_m(t)$ so that it can be extended to the whole interval $[0, T]$, $T > 0$.

**Priori Estimates**: We replace $v = u_m'(t)$ in the approximate equation (8) and we get

$$(u_m'(t), u_m'(t)) - \langle \Delta_p u_m(t), u_m'(t)\rangle_p - (\Delta u_m(t), u_m'(t)) + (g * \Delta u_m(t), u_m'(t)) = 0 \qquad (9)$$

Let $\theta \in D(0, t_m)$. We denote by $\langle \cdot\,, \cdot \rangle$ the duality pairing between $D'$ and $D$. So we have,

$$\langle (u_m''(t), u_m'(t)), \theta\rangle = \left\langle \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m'(t)|_2^2, \theta \right\rangle \qquad (10)$$

$$\langle\langle -\Delta_p u_m(t), u_m'(t)\rangle_p, \theta\rangle = \left\langle \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_m(t)\|_p^p, \theta \right\rangle \qquad (11)$$

$$\langle (-\Delta u_m(t), u_m'(t)), \theta\rangle = \left\langle \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u_m(t)|_2^2, \theta \right\rangle \tag{12}$$

Now, note that

$$(g * \Delta u_m(t), u_m'(t)) = -\left((g * \nabla u_m)(t), \nabla u_m'(t)\right).$$

By Lemma 3.2 we have

$$\begin{aligned} 2\left((g * \nabla u_m)(t), \nabla u_m'(t)\right) = & \int_\Omega (g' \Box \nabla u_m)(t)\,\mathrm{d}x - g(t)\int_\Omega |\nabla u_m(t)|^2\,\mathrm{d}x \\ & - \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\left[(g\Box\nabla u_m)(t) - \left(\int_0^t g(s)\,\mathrm{d}s\right)|\nabla u_(t)|^2\right]\mathrm{d}x. \end{aligned}$$

Then,

$$\begin{aligned} \langle (g * \Delta u_m(t), u_m'(t)), \theta\rangle = & \left\langle -\frac{1}{2}\int_\Omega (g' \Box \nabla u_m)(t)\,\mathrm{d}x + \frac{1}{2}g(t)|\nabla u_m(t)|_2^2 \right. \\ & \left. +\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (g\Box\nabla u_m)(t)\,\mathrm{d}x - \left(\int_0^t g(s)\,\mathrm{d}s\right)|\nabla u_m(t)|_2^2, \theta\right\rangle. \end{aligned} \tag{13}$$

Replacing (10), (11), (12), (13) in (9) we obtain in $D'(0, t_m)$

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}|u_m'(t)|_2^2 + \frac{1}{p}\|\nabla u_m(t)\|_p^p + \frac{1}{2}(g\Box\nabla u_m)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla u_m(t)|_2^2\right\} \\ = & \frac{1}{2}\int_\Omega (g'\Box\nabla u_m)(t)\,\mathrm{d}x - \frac{1}{2}g(t)|\nabla u_m(t)|_2^2 \end{aligned} \tag{14}$$

The approximate energy

$$E_m(t) = \frac{1}{2}|u_m'(t)|_2^2 + \frac{1}{p}\|\nabla u_m(t)\|_p^p + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla u_m(t)|_2^2 + \frac{1}{2}(g\Box\nabla u_m)(t)$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m(t) \le -\frac{c_1}{2}\int_\Omega (g\Box\nabla u_m)(t)\,\mathrm{d}x - \frac{1}{2}g(t)|\nabla u_m(t)|_2^2.$$

Then $E_m(t) \le E_m(0)$. Due to convergence of initial data, there exists a constant $c > 0$ independent of $t$ and $m$ such that $E_m(t) \le c$. With this estimate we can extend the aproximate solutions $u_m(t)$ to the interval $[0, T]$, see [25], and we have

$$u_m(t) \text{is bounded in} L^\infty(0, T; W_0^{1,p}(\Omega)), \tag{15}$$

$$u_m'(t) \text{is bounded in} L^\infty(0, T; L^2(\Omega)), \tag{16}$$

$$-\Delta_p u_m(t) \text{is bounded in} L^\infty(0, T; W^{-1,p'}(\Omega)). \tag{17}$$

From (15) and Lemma 3.1 we deduce

$$(g * \nabla u_m)(t) \text{ is bounded in } L^\infty(0, T; L^2(\Omega)). \tag{18}$$

**Passage to the Limit**: From (15), (16), (18) going to the subsequence if necessary, there exists $u$ such that

$$u_m \rightharpoonup u \ \text{ weakly star in } L^\infty(0, T; W_0^{1,p}(\Omega)) \tag{19}$$

$$u'_m \rightharpoonup u' \text{ weakly star in } L^\infty(0,T;L^2(\Omega)) \tag{20}$$

$$g * \nabla u_m \rightharpoonup g * \nabla u \text{ weakly star in } L^\infty(0,T;L^2(\Omega)) \tag{21}$$

and in view of (17) there exists $\mathcal{X}$ such that

$$-\Delta_p u_m(t) \to \mathcal{X} \text{ weakly in } L^\infty(0,T;W^{-1,p'}(\Omega)). \tag{22}$$

With these convergence we can pass to the limit in the approximate equation (8) see [26, 27], and then

$$\frac{\mathrm{d}}{\mathrm{d}t}(u'(t),v) + \langle \mathcal{X}(t), v\rangle_p + (-\Delta u(t), v) + ((g*\nabla u)(t), v) = 0,$$

for all $v \in W_0^{1,p}(\Omega)$ in the sense of distributions.
For $x, y \in \mathbb{R}$ and $p \geq 2$, consider the elementary inequalities

$$\left| |x|^{\frac{p-2}{2}}x - |y|^{\frac{p-2}{2}}y \right| \leq C\left( |x|^{\frac{p-2}{2}} + |y|^{\frac{p-2}{2}} \right)|x-y|, \tag{23}$$

$$\left| |x|^{p-2}x - |y|^{p-2}y \right| \leq C\left( |x|^{\frac{p-2}{2}} + |y|^{\frac{p-2}{2}} \right)\left| |x|^{\frac{p-2}{2}}x - |y|^{\frac{p-2}{2}}y \right|. \tag{24}$$

The inequality (23) is a consequence of the mean value theorem and (24) can be found in [28]. As in [29] applying (23), (24) and Hölder generalized inequality with

$$\frac{p-2}{4p} + \frac{p-2}{4p} + \frac{1}{2} + \frac{1}{p} = 1$$

we deduce for all $v \in W_0^{1,p}(\Omega)$

$$\left| \int_0^T \langle -\Delta_p u_m(t), v\rangle_p - \langle -\Delta_p u(t), v\rangle_p \,\mathrm{d}t \right| \leq c \int_0^T |\nabla u_m(t) - \nabla u(t)|_2 \,\mathrm{d}t.$$

Now we are going to obtain an estimate for $u''_m(t)$. Since our Galerkin basis was taken in the Hilbert space $L^2(\Omega)$ we can use the standard projection arguments as described in Lions [26]. Then from the approximate equation and the estimates (15)-(17) we get

$$u''_m(t) \quad \text{is bounded in} \quad L^\infty(0,T;W^{-1,q}(\Omega)). \tag{25}$$

Applying the Lions-Aubin compactness we get from (19), (20) and (25),

$$u_m(t) \to u(t) \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{26}$$

$$u'_m(t) \to u'(t) \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{27}$$

Using (26) we get that $u_m(t) \to u(t)$ almost everewhere in $\Omega \times (0,T)$ and we have,

$$-\Delta_p u_m(t) \to -\Delta_p u(t) \text{ weakly in } L^\infty(0,T;W^{-1,p'}(\Omega)). \tag{28}$$

From (22), (28) and uniqueness of the limit we conclude that $\mathcal{X}(t) = -\Delta_p u(t)$. The verification of the initial data is a routine procedure. The prove of existence is complete.

□

**5.2. Uniqueness.** Let $u$ and $v$ be solutions of (2)-(4) such that

$$u(x,0) = u_0 = v(x,0) \quad \text{and} \quad u_t(x,0) = u_1 = v_t(x,0).$$

Denoting $w = u - v$ we have

$$u_{tt} - \Delta w = \Delta_p u - \Delta_p v - g * \Delta w, \quad \text{in} \quad \Omega \times [0,\infty), \tag{29}$$

$$w(x,0)=0, \; u_t(x,0) = 0, \; x \in \Omega, \tag{30}$$

$$w(x,t)=0 \quad \text{on} \quad \partial\Omega \times [0,\infty). \tag{31}$$

We will use the Višik-Ladyenskaya method [30]. Consider for each $\eta \in [0,T]$ the following function

$$\psi(x,t) = \begin{cases} -\displaystyle\int_t^\eta w(x,\xi)\,\mathrm{d}\xi &, \quad 0 \le t < \eta, \\ \quad 0 &, \quad \eta \le t \le T. \end{cases} \tag{32}$$

Then,

$$\psi_t(x,t) = \begin{cases} w(x,t) &, \quad 0 \le t < \eta, \\ \quad 0 &, \quad \eta \le t \le T. \end{cases} \tag{33}$$

As $w \in L^\infty(0,T;W_0^{1,p}(\Omega))$, $w_t \in L^\infty(0,T;L^2(\Omega))$ we have

$$\psi, \; \psi_t \in L^\infty(0,T;L^2(\Omega)). \tag{34}$$

Mutiplying (29) by $\psi$ and performing integration on $\Omega$

$$(w_{tt},\psi) + (\nabla w, \nabla\psi) = \langle \Delta_p u - \Delta_p v, \psi\rangle_p - (g * \Delta w, \psi).$$

Integrating in $[0,\eta]$ and taking into account that $\psi(x,t) \equiv 0$ for all $t \in [\eta,T]$, we have

$$\int_0^\eta (w_{tt},\psi)\,\mathrm{d}t + \int_0^\eta (\nabla w, \nabla\psi)\,\mathrm{d}t = \int_0^\eta \langle \Delta_p u - \Delta_p v, \psi\rangle_p\,\mathrm{d}t - \int_0^\eta (g * \Delta w), \psi)\,\mathrm{d}t.$$

As $\psi(\eta) = w(0) = 0$ we get

$$-\int_0^\eta (w_t,\psi_t)\,\mathrm{d}t + \int_0^\eta (\nabla w, \nabla\psi)\,\mathrm{d}t = \int_0^\eta \langle \Delta_p u - \Delta_p v, \psi\rangle_p\,\mathrm{d}t - \int_0^\eta ((g * \Delta w)(t), \psi)\,\mathrm{d}t.$$

From (31), (32) and (33)

$$-\int_0^\eta (w_t,w)\,\mathrm{d}t + \int_0^\eta (\nabla \psi_t, \nabla\psi)\,\mathrm{d}t = \int_0^\eta \langle \Delta_p u - \Delta_p v, \psi\rangle_p\,\mathrm{d}t - \int_0^\eta (g * \Delta w, \psi)\,\mathrm{d}t.$$

That is

$$-\frac{1}{2}\int_0^\eta \frac{\mathrm{d}}{\mathrm{d}t}|w|_2^2\,\mathrm{d}t + \frac{1}{2}\int_0^\eta \frac{\mathrm{d}}{\mathrm{d}t}|\nabla\psi|_2^2\,\mathrm{d}t = \int_0^\eta \langle \Delta_p u - \Delta_p v, \psi\rangle_p\,\mathrm{d}t - \int_0^\eta ((g * \Delta w)(t), \psi)\,\mathrm{d}t,$$

that implies

$$-\frac{1}{2}|w(\eta)|_2^2 - \frac{1}{2}|\nabla\psi(0)|_2^2 \le \int_0^\eta \langle \Delta_p u - \Delta_p v, \psi\rangle_p\,\mathrm{d}t - \int_0^\eta ((g * \Delta w)(t), \psi)\,\mathrm{d}t \tag{35}$$

As before, applying (23), (24) and Hölder generalized inequality with

$$\frac{p-2}{4p}+\frac{p-2}{4p}+\frac{1}{p}+\frac{1}{2}=1$$

we obtain

$$|\langle \Delta_p u-\Delta_p v,\psi\rangle_p|\leq c|\nabla\psi|_2. \tag{36}$$

From (34) and continuous and dense injection (7) we deduce that $g*\Delta w\in L^2(0,T;L^2(\Omega))$. Then

$$\left|\int_\Omega (g*\Delta w)(t)\psi\,\mathrm{d}x\right|\leq\left(\int_\Omega |g*\Delta w|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |\psi|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\leq c|\psi|_2. \tag{37}$$

From (35), (36), (37), Poincarè and Cauchy-Schwarz inequalities we deduce

$$\frac{1}{2}|w|_2^2+\frac{1}{2}|\nabla\psi(0)|_2^2\leq c\int_0^\eta |\nabla\psi|_2^2\,\mathrm{d}t. \tag{38}$$

Now we introduce $w_1(x,t)=\displaystyle\int_0^t w(x,\xi)\,\mathrm{d}\xi$, for all $t\in[0,\xi)$ we have

$$\psi(x,t)=-\int_0^\eta w(x,\xi)\,\mathrm{d}\xi=-\int_0^\eta w(x,\xi)\,\mathrm{d}\xi+\int_0^t w(x,\xi)\,\mathrm{d}\xi=w_1(x,t)-w_1(x,\eta), \tag{39}$$

and then

$$\psi(x,0)=w_1(x,0)-w_1(x,\eta)=-w_1(x,\eta). \tag{40}$$

From (38), (39) and (40) we obtain

$$\frac{1}{2}|\nabla w_1|_2^2\leq c\int_0^\eta |\nabla w_1|_2^2\,\mathrm{d}t.$$

By Gronwall's inequality we conclude $|\nabla w_1|_2^2\leq 0$. By (39) we deduce that $\nabla\psi=0$ in $L^2(\Omega)$ for all $t\in[0,T]$. Finally follows from (38) that $|w|_2^2\leq 0$ and then $u=v$ in $L^2(\Omega)$ for all $t\in[0,T]$.

## 6. Asymptotic Behaviour

**Theorem 6.1.** *Let $u$ be a solution of (2)-(4) with initial data $u_0\in V$, $u_1\in L^2(\Omega)$. For $\phi\colon\mathbb{R}_+\to\mathbb{R}_+$ a increasing $C^2$ function such that $\phi(0)=0$ and $\phi(t)\xrightarrow{t\to+\infty}+\infty$ we have for $q>0$*

$$E(t)\leq cE(0)(1+\phi(t))^{\frac{-1}{q}},\ \forall\ t>0 \tag{41}$$

*where $c$ is a positive constant independent of the initial energy $E(0)$.*

*Proof.* We will use Lemma 3.3 due to P. Martinez [16] based on a new inequality that generalizes a result of Haraux [17]. For the goal we start the proof of (41) multiplying (2) by $E^q(t)\phi'(t)u$ and so we set

$$\int_0^T E^q\phi'\int_\Omega u(u_{tt}-\Delta_p u-\Delta u+g*\Delta u)\,\mathrm{d}x\,\mathrm{d}t=0,$$

from where we obtain by straight calculations

$$
\begin{aligned}
2\int_S^T E^{q+1}\phi'\,\mathrm{d}t \quad \le \quad & -\left[E^q\phi'\int_\Omega uu_t\,\mathrm{d}x\right]_S^T \\
& +4\int_S^T\left[\left(qE'E^{q-1}\phi'+E^q\phi''\right)\int_\Omega uu_t\,\mathrm{d}x\right]\mathrm{d}t \\
& +4\int_S^T\left[E^q\phi'\frac{1}{2}\int_\Omega|u_t|^2\,\mathrm{d}x\right]\mathrm{d}t+\int_S^T\left[E^q\phi'\frac{1}{2}\int_\Omega g\Box\nabla u\,\mathrm{d}x\right]\mathrm{d}t \\
& +\int_S^T\left[E^q\phi'\frac{1}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2\,\mathrm{d}s\,\mathrm{d}x\right]\mathrm{d}t \\
& +\int_S^T\left[E^q\phi\frac{1}{2}\int_\Omega|\nabla u|^2\,\mathrm{d}x\int_0^T g(t-s)\,\mathrm{d}s\right]\mathrm{d}t. \qquad (42)
\end{aligned}
$$

In the stability set $V$ we have

$$
\frac{1}{2}\int_0^t g(t-s)|\nabla u(s)|^2\,\mathrm{d}s\le\frac{1}{p}\|\nabla u(s)\|_p^p. \qquad (43)
$$

Replacing (43) in (42) we obtain

$$
\begin{aligned}
\int_S^T E^{q+1}\phi'\,\mathrm{d}t \quad \le \quad & -\left[E^q\phi'\int_\Omega uu_t\,\mathrm{d}x\right]_S^T+\int_0^T\left[\left(qE'E^{q-1}\phi'+E^q\phi''\right)\int_\Omega uu_t\,\mathrm{d}x\right]\mathrm{d}t \\
& +\frac{3}{2}\int_S^T E^q\phi'\int_\Omega|u_t|^2\,\mathrm{d}x\,\mathrm{d}t. \qquad (44)
\end{aligned}
$$

Now, we will estimate each term on the right side of (44). Applying the same argument as in [6] we deduce

$$
\left|\left[E^q\phi'\int_\Omega uu_t\,\mathrm{d}x\right]_S^T\right|\le cE(s),\ \forall\ t\ge S \qquad (45)
$$

and

$$
\left|\int_S^T\left[\left(qE'E^{q-1}\phi'+E^q\phi''\right)\int_\Omega uu_t\,\mathrm{d}x\right]\mathrm{d}t\right|\le cE(s),\ \forall\ t\ge S. \qquad (46)
$$

The estimate of $\int_S^T E^q\phi'\int_\Omega|u_t|^2\,\mathrm{d}x\,\mathrm{d}t$ is quite delicate. Let $\sigma\colon\mathbb{R}_+\to\mathbb{R}_+$ be a strictly positive function such that $\int_0^\infty\sigma(t)\,\mathrm{d}\tau=+\infty$. $\phi(t)=\int_0^t\sigma(\tau)\,\mathrm{d}\tau$ satisfies $\phi(0)=0$ and $\phi(t)\xrightarrow{t\to+\infty}+\infty$. Consider $\rho(t,u)\le-E'(t)$.

According to the [6] for all $0 < S < T$ ande $l < m+1$

$$\begin{aligned}\int_S^T E^q\phi'\int_\Omega |u_t|^l\,\mathrm{d}x\,\mathrm{d}t &\leq c\int_S^T E^q\phi'\int_\Omega \frac{1}{\sigma(t)}u_t\rho(t,u)\,\mathrm{d}x\,\mathrm{d}t\\ &+c'\int_S^T E^q\phi'\int_\Omega\left(\frac{1}{\sigma(t)}u_t\rho(t,u)\right)^{\frac{l}{m+1}}\,\mathrm{d}x\,\mathrm{d}t\\ &\leq c\int_S^T E^q\frac{\phi'}{\sigma(t)}(-E')\int_\Omega |u'|\,\mathrm{d}x\,\mathrm{d}t\\ &+c'\int_S^T E^q\phi'\sigma^{-\frac{l}{m+1}}(t)(-E')^{\frac{l}{m+1}}\int_\Omega |u'|^{\frac{l}{m+1}}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Applying Hölder inequality, using $l < m+1$, $|u'|_2^2 < c$ we get

$$\begin{aligned}\int_S^T E^q\phi'\int_\Omega |u_t|^l\,\mathrm{d}x\,\mathrm{d}t &\leq c\int_S^T E^q\frac{\phi'}{\sigma(t)}(-E')\,\mathrm{d}t\\ &+c'\int_S^T E^q\phi'^{\frac{m+1-l}{m+1}}\left(\frac{\phi'}{\sigma(t)}\right)^{\frac{l}{m+1}}(-E')^{\frac{l}{m+1}}\,\mathrm{d}t.\end{aligned}$$

For fix and arbitrarily small $\varepsilon > 0$ (to be chosen later). By applying Young's inequality $\dfrac{1}{\frac{m+1}{m+1-l}}+\dfrac{1}{\frac{m+1}{l}}=1$ we obtain

$$\begin{aligned}\int_S^T E^q\phi'\int_\Omega |u'|^l\,\mathrm{d}x\,\mathrm{d}t &\leq c\int_S^T E^q\frac{\phi'}{\sigma(t)}(-E')\,\mathrm{d}t\\ &+c'\frac{m+1-l}{m+1}\varepsilon^{\frac{m+1}{m+1-l}}\int_S^T E^{q\frac{m+1}{m+1-l}}\phi'\,\mathrm{d}t\\ &+c'\frac{l}{m+1}\int_\Omega(-E')\frac{\phi'}{\sigma(t)}\varepsilon^{-\frac{m+1}{l}}\,\mathrm{d}t.\end{aligned}$$

From where follows

$$\begin{aligned}\int_S^T E^q\phi'\int_\Omega |u_t|^l\,\mathrm{d}x\,\mathrm{d}t &\leq cE^q(s)+c'\frac{m+1-l}{m+1}\varepsilon^{\frac{m+1}{m+1-l}}\int_S^T E^{q\frac{m+1}{m+1-l}}\phi'\,\mathrm{d}t\\ &+c'\frac{l}{m+1}\varepsilon^{-\frac{m+1}{l}}E(s).\end{aligned}$$

Making $l=2$, $\rho(t,u)=\dfrac{1}{2}\displaystyle\int_\Omega (g\Box\nabla u)(t)\,\mathrm{d}x$ e choosing $q$ such that

$$\frac{m+1}{m+1-l}=\frac{q+1}{q}$$

we obtain

$$\int_S^T E^q\phi'\int_\Omega |u_t|^2\,\mathrm{d}x\,\mathrm{d}t \leq cE(s)+c\frac{m-2}{m+1}\varepsilon^{\frac{m+1}{m-1}}\int_S^T E^{q+1}\phi'\,\mathrm{d}t+\frac{2c}{m+1}\varepsilon^{-\frac{m+1}{2}}E(s) \tag{47}$$

Now we deduce from (44), (45), (46) and (47)

$$\left(1-c\frac{m-1}{m+1}\varepsilon^{\frac{m+1}{m-1}}\right)\int_S^T E^{q+1}\phi'\,\mathrm{d}t \leq c(\varepsilon)E(s).$$

Finally, choosing $\varepsilon$ small enough we concludes

$$\int_S^T E^{q+1}\phi'\,\mathrm{d}t \leq cE(s),\ q>0$$

and the proof is complete. $\square$

## Concluding remarks

When $p=2$ is well known that the equation (2) describes a homogeneous and isotropic viscoelastic solid and the genuine memory $g*\Delta u$ induces a damping mechanism so the asymptotic stability is to be expected. For instance, for the nonhomogeneous problem with function $f$ independent of time and source term the existence of a global attractor was proved in [31]. The problem with supercritical source and damping terms was studied in [32]. Employing the theory of monotone operators and nonlinear semigroups, combined with energy methods was established the existence of a unique local weak solution in the finite energy space. As follow-up work, recently in [33] was considered supercritical nonlinearities and was studied blow-up of solutions when the source is stronger than dissipation. For The case $p>2$ the nonlinear equation (2) leads to a problem not previously considered. The highlight here was to prove the existence of solution and energy decay in the appropriate set of stability created from the Nehari manifold.

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## Competing Interests

The authors declare that they have no competing interests.

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