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Numerical Fractional Differentiation: Stability Estimate and Regularization

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Abstract

It is well known that the problem of fractional differentiation n is an ill-posed problem. So far there exists many approximation methods for solving this problem. In this paper we prove a stability estimate for a problem of fractional differentiation. Based on the obtained stability estimate, we present a Tikhonov regularization method and obtain the error estimate. According to the optimality theory of regularization, the error estimates are order optimal. Numerical experiment shows that the regularization works well.

Keywords: *Fractional differentiation, ill-posed problems, Tikhonov regularization, stability, estimate, error estimate.*

1 Introduction

The basic theory and applications of fractional differential equations are covered extensively in the literature [1,2] etc. However, the topics related to ill-posed inverse problems are few, see [3,4,5].

Fractional differentiation problems arises in many contexts and have important applications in science and engineering [6,7,8]. In this paper, we consider the problem of fractional differentiation in $L^2(R)$. We want to compute

$$g(x) := D^{\beta} f(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_{-\infty}^{x} \frac{f(t)}{(x-t)^{\beta}} dt \tag{1.1}$$

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which is called the β -th order Riemann-Liouville fractional derivative of the function $f(x)$ with $\beta \in (0,1]$,where $\Gamma(\cdot)$ is the Euler Gamma function. If there exists noise in $f(x)$, then the problem $g(x) = D^{\beta} f(x)$ for solving $g(x)$ is an ill-posed problem. The problem of fractional differentiation amounts to the problem of solving the integral equation of the fist kind [9,10,11].

$$(A_{\beta u})(x) := \frac{1}{\Gamma(\beta)} x \int_{-\infty} \frac{g(t)}{(x-t)^{1-\beta}} dt = f(x) \tag{1.2}$$

This problem have been studied by many authors [12,13,14,15], and a large number of different solution methods has been proposed. For references we refer the reader to [12,13,14,15,16,17]. Finite difference approaches for numerical differentiation have been used, for example, in[19,20]. However, these approaches require a knowledge of a bound of the second or third derivatives of the function under consideration that are not always available. Furthermore, there exist infinitely many functions that do not have bounded second derivatives at all. The same situation occurs also in numerical fractional differentiation [3]. one requires a high smoothness of the functions under consideration that does not always exist.

Assume that

$$\left\| f(\cdot) - f^{\delta}(\cdot) \right\| \le \delta, \tag{1.3}$$

where $\|\cdot\|$ denotes the L^2 -norm, and the constant $\delta > 0$ represents the noise level.
Let

$$\hat{s}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s(x) e^{-i\xi x} dx \tag{1.4}$$

be the Fourier transform of the function $s(x) \in L^2(R)$. The corresponding inverse Fourier transform of the function $\hat{s}(\xi)$ is

$$s(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{s}(\xi) e^{i\xi x} d\xi \tag{1.5}$$

In the frequency domain, we have

$$(i\xi)^{\beta} \hat{f}(\xi) = \hat{g}(\xi) \tag{1.6}$$

From (1.6), we can formulate the problem as an operator equation

$$\hat{A}\hat{g}(\xi) = \hat{f}(\xi) \tag{1.7}$$

where $\hat{A} = (i\xi)^{-\beta} : L^2(R) \to L^2(R)$ is a self-adjoint multiplication operator.

Since the factor $(i\xi)^{\beta} \to \infty$ in (1.6) as $\xi \to \infty$, it is easy to see that the ill-posedness of Problem (1.1).

However, most of the literature is devoted to the regularization method and error estimate. The conditional stability estimate is not clear. In this paper, our task is to establish a stability estimate. For error estimate of regularization, the method in this study is different from the existing literature, e.g., [13].

1. (1)In [13], the authors got the error estimate of molliffication regularization by direct calculation the difference between the regularized solution and exact one.
2. (2) In this study, we use the obtained stability estimate and we only need to estimate one term.

The advantage of our method for obtaining error estimate over the method in [13] lie in:

1. we can obtain stability estimate which is an important issue for study on ill-posed problems;
2. fewer quantities are needed to be estimated, thus, our method is easier to obtain error estimate;

The paper is organized as follows. In Section 2, a conditional stability estimate for problem (1.1) is proved; in Section 3, the error estimate for a Tikhonov regularization method is given.

2 A Conditional Stability Estimate

A conditional stability estimate for ill-posed problems tells how much any two solutions differ from each other when some error exists. Since problem (1.1) is linear, we only need to derive the stability estimate on the solution near zero between the zero one.

For problem (1.1), we assume that there exists a-priori bound:

$$\|g(\cdot)\|_p \le E, \tag{2.1}$$

Where $\|\cdot\|_p$ denotes the norm of Sobolev space $H^p(R)$ and $p > 0$, E is a positive constant. Thus, we can establish the stability estimate for problem (1.1).

Theorem 2.1: Suppose that $g(x)$ is the solution of problem (1.1), and (2.1) is satisfied, then the following estimate holds:

$$\|g(\cdot)\| \le \|f(\cdot)\| + \|g(\cdot)\|_p^{\frac{\beta}{p+\beta}} \|f(\cdot)\|^{\frac{p}{p+\beta}}. \tag{2.2}$$

Proof: Since $\hat{g}(\xi) = (i\xi)^\beta \hat{f}(\xi)$, by Parseval identity we have

$$\|g(\cdot)\|^2 = \|\hat{g}(\cdot)\|^2 = \int_{|\xi|\le 1} |\xi|^{2\beta} \left|\hat{f}(\xi)\right|^2 d\xi + \int_{|\xi|\ge 1} |\hat{g}(\xi)|^2 d\xi := B_1 + B_2.$$

Obviously,

$$B_1 \le \|f(\cdot)\|^2$$

Further by Holder inequality,

$$\begin{aligned}
B_2 &= \int_{|\xi|\ge 1} |\hat{g}(\xi)|^2 d\xi \\
&= \int_{|\xi|\ge 1} [(1+\xi^2)^p |\hat{g}(\xi)|^2]^{\frac{\beta}{p+\beta}} [(1+\xi^2)^{-\beta} |\hat{g}(\xi)|^2]^{\frac{p}{p+\beta}} d\xi \\
&\le (\int_{|\xi|\ge 1} (1+\xi^2)^p |\hat{g}(\xi)|^2 d\xi)^{\frac{\beta}{p+\beta}} (\int_{|\xi|\ge 1} (1+\xi^2)^{-\beta} |\hat{g}(\xi)|^2 d\xi)^{\frac{p}{p+\beta}} \\
&\le \|g(\cdot)\|_p^{\frac{2\beta}{p+\beta}} \cdot (\int_{|\xi|\ge 1} (1+\xi^2)^{-\beta} \left|(i\xi)^\beta \hat{f}(\xi)\right|^2 d\xi)^{\frac{p}{p+\beta}} \\
&\le \|g(\cdot)\|_p^{\frac{2\beta}{p+\beta}} \cdot (\sup_{|\xi|\ge 1} \frac{\xi^{2\beta}}{(1+\xi^2)^\beta} \int_{-\infty}^{+\infty} \left|\hat{f}(\xi)\right|^2 d\xi)^{\frac{p}{p+\beta}} \\
&\le \|g(\cdot)\|_p^{\frac{2\beta}{p+\beta}} \cdot \|f(\cdot)\|^{\frac{2p}{p+\beta}}.
\end{aligned}$$

Hence we have (2.2).

Remark 2.1: For given two functions $f_1(\cdot)$ and $f_2(\cdot)$, let $g_1(\cdot)$ and $g_2(\cdot)$ be the corresponding solutions, respectively, then

$$\|g_1(\cdot) - g_2(\cdot)\| \le \|f_1(\cdot) - f_2(\cdot)\| + \|g_1(\cdot) - g_2(\cdot)\|_p^{\frac{\beta}{p+\beta}} \|f_1(\cdot) - f_2(\cdot)\|^{\frac{p}{p+\beta}} \tag{2.3}$$

3 Tikhonov Regularization and Error Estimate

Now we focus on the regularization problem. Assume that the noisy data $f_\delta(\cdot)$ satisfies

$$\|f(\cdot) - f_\delta(\cdot)\| \le \delta \tag{3.1}$$

where $\delta > 0$ is the noise level.

Here we use Tikhonov regularization method [16]. The method consists of looking for the solution for problem (1.1), and minimizes the quadratic functional:

$$\|f(\cdot) - f_\delta\|^2 + \alpha\|g(\cdot)\|_p^{\ 2} \tag{3.2}$$

Where $\alpha = \delta^2 / E^2$.By Parseval's identity and (1.6) applied to $f(\cdot)$, the variational problem becomes

$$\left\|(i\xi)^{-\beta}\hat{g}(\cdot) - \hat{f}_\delta(\cdot)\right\|^2 + \alpha\left\|(1+\xi^2)^{p/2}\hat{g}(\cdot)\right\|^2 \tag{3.3}$$

Let $\hat{g}_\delta^{\ \alpha}(\cdot)$ be the solution of above problem, then it satisfies the Euler equation

$$(|\xi|^{-2\beta} + \alpha(1+\xi^2)^p)\hat{g}_\delta^{\ \alpha}(\xi) = (i\xi)^{-\beta}\hat{f}_\delta(\xi) \tag{3.4}$$

The Tikhonov regularization solution $\hat{g}_\delta^{\ \alpha}(\cdot)$ in frequency domain can be given

$$\hat{g}_\delta^{\ \alpha}(\xi) = \frac{(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\hat{f}_\delta(\xi) \tag{3.5}$$

Now we will use the stability estimate (2.3) to derive the error estimate between the regularization solution and the exact one. Via Parseval's identity, we have

$$\begin{aligned}
\left\|\hat{g}_\delta^{\ \alpha}(\cdot) - \hat{g}(\cdot)\right\|_p &= \left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\frac{(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\hat{f}_\delta - (i\xi)^\beta\hat{f}\right|^2 d\xi\right)^{1/2} \\
&\le \left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\frac{(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\hat{f}_\delta - \frac{(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\hat{f}\right|^2 d\xi\right)^{1/2} \\
&+ \left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\frac{(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\hat{f} - (i\xi)^\beta\hat{f}\right|^2 d\xi\right)^{1/2} \\
&\le \sup_{\xi\in R}\left|\frac{(1+\xi^2)^{p/2}(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\right|\left\|\hat{f}_\delta - \hat{f}\right\| + \sup_{\xi\in R}\left|\frac{\alpha(1+\xi^2)^p|\xi|^{2\beta}}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\right|\left\|\hat{g}(\cdot)\right\|_p
\end{aligned}$$

$$\left\|\hat{g}_\delta{}^\alpha(\cdot)-\hat{g}(\cdot)\right\|_p=\left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\frac{(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\hat{f}_\delta-(i\xi)^\beta\hat{f}\right|^2 d\xi\right)^{1/2}$$

$$\leq\left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\frac{(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\hat{f}_\delta-\frac{(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\hat{f}\right|^2 d\xi\right)^{1/2}$$

$$+\left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\frac{(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\hat{f}-(i\xi)^\beta\hat{f}\right|^2 d\xi\right)^{1/2}$$

$$\leq\sup_{\xi\in R}\left|\frac{(1+\xi^2)^{p/2}(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\right|\left\|\hat{f}_\delta-\hat{f}\right\|+\sup_{\xi\in R}\left|\frac{\alpha(1+\xi^2)^p|\xi|^{2\beta}}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\right|\left\|\hat{g}(\cdot)\right\|_p$$

Since $h(\eta):=\eta(1+\eta^2)^{-1},\eta>0$ has a maximum $1/2$,

$$\sup_{\xi\in R}\left|\frac{(1+\xi^2)^{p/2}(i\xi)^\beta}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\right|\leq\frac{1}{2}(\sqrt{\alpha})^{-1}=\frac{1}{2}\frac{E}{\delta}$$

$$\sup_{\xi\in R}\left|\frac{(1+\xi^2)^p|\xi|^{2\beta}}{1+\alpha(1+\xi^2)^p|\xi|^{2\beta}}\right|\leq 1$$

Thus

$$\left\|\hat{f}_\delta{}^\alpha(\cdot)-\hat{f}(\cdot)\right\|_p\leq\frac{3}{2}E. \tag{3.6}$$

Hence according to (2.3) we have the error estimate

$$\left\|g_\delta{}^\alpha(\cdot)-g(\cdot)\right\|\leq\delta+\frac{3}{2}E^{\frac{\beta}{p+\beta}}\delta^{\frac{p}{p+\beta}} \tag{3.7}$$

Now summarize what we have.

Theorem 3.1: Suppose that $g(x)$ is the exact solution for problem (1.1) with exact data $f(x)$, (3.5) is the Tikhonov regularization solution with noisy data $f_\delta(x)$, (2.1) and (3.1) hold, then we have

$$\left\| g_\delta{}^\alpha(\cdot) - g(\cdot) \right\| \le \delta + \frac{3}{2} E^{\frac{\beta}{p+\beta}} \delta^{\frac{p}{p+\beta}} \tag{3.8}$$

Remark 3.1: Now we seek a solution in frequency domain for problem (1.1) in $L^2[-\xi_{\max}, \xi_{\max}]$, where $\xi_{\max}$ is a positive number which plays a role of regularization parameter. This is the so-called Fourier cut-off method. Similar to Tikhonov regularization, we can easily obtain the error estimate for Fourier cut-off method.

From Theorem 3.1, we find that $g_\delta{}^\alpha$ is an approximation of exact solution g. The approximation error depends continuously on the measurement error.

4 Numerical Tests

Suppose that the sequence $\{f_j\}_{j=1}^{m-1}$ represents samples from $f(x)$ on an equidistant grid $0 = x_0 < \cdots < x_{m-1} = 1$, then we add a perturbation to each data, and obtain the perturbation data

$$f_\delta = f + \varepsilon \cdot randn(size(f)) \tag{4.1}$$

Where

$$f = (f(x_0), \cdots f(x_{m-1}))^T, x_j = j\Delta x,$$
$$\Delta x = \frac{1}{m-1}, j = 0, \cdots m-1,$$
$$\delta = \| f_\delta - f \|_{l^2} = \sqrt{\frac{1}{m-1} \sum_{j=0}^{m-1} (f_\delta(x_j) - f(x_j))^2}$$

The function "$randn(\cdot)$" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$, "$randn(size(f))$" returns an array of random entries that is the same size as f.

We now give a simple description of numerical implementation.

Step 1: taking the Fast Fourier Transform (*FFT*) for the vector f_δ.
Step 2: computing the vector.

$$\{\frac{(i\xi_l)^\beta}{1+(\mu|\xi_l|)^{2\alpha}}\hat{f}_\delta(\xi_l)\}_{l=-m/2}^{m/2+1}, \tag{4.2}$$

Where $i=\sqrt{-1}, \xi_l = 2\pi l.$

Step 3: taking the inverse *FFT* for the vector in (4.2) and get $R(f_\delta{}^\beta)$.

When using the *FFT* algorithm we implicitly assume that the vector f_δ represents a periodic function. This is not realistic in our application, and thus we need to modify the algorithm. A discussion on how to make the function periodic can be found in [20]. Note also that the error estimates of section 3 contain norms in $L^2(R)$, and the numerical experiments here are performed with a finite interval. However, the method for selecting μ works well, also in the case where the norms are computed for a finite interval. In our computations, we always take $m=300$ (If we take $m=100,200,\cdots 400,$ we can also obtain a satisfactory result). The derivative errors are measured by the weighted l_2-norms defined as follows:

$$E(R(f_\delta{}^\alpha)) = \sqrt{\frac{1}{m-1}\sum_{j=0}^{m-1}(f^\alpha(x_j) - R(f_\delta{}^\alpha)(x_j))^2} \tag{4.3}$$

Example: We want to reconstruct the $D^{1/2}f(x)$,where $f(x)=\exp(-x^2/2)$ .In numerical experiment, we add a random noise to the $f(x)$. The numerical results are shown as following.

Figure 1: we take the optimal regularization parameter $\mu = 0.01$;
Figure 2: we take the optimal regularization parameter $\mu = 0.001$.

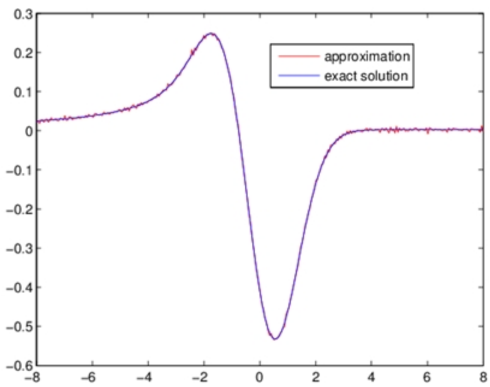

Figure 1: $\delta = 1*10^{-3}, \mu = 0.01.$

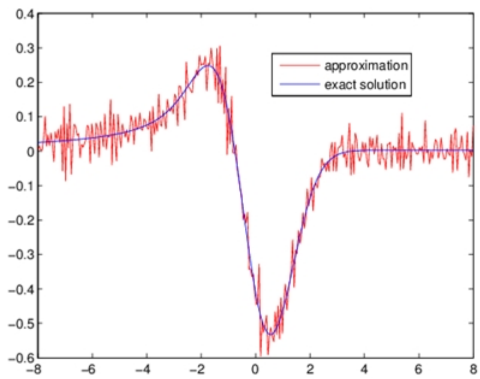

Figure 2: $\delta = 1*10^{-3}, \mu = 0.001.$

5 Concluding Remark

The issue of conditional stability on inverse ill-posed problems is very important. In this paper, we give an 'optimal' stability estimate. Based on the obtained estimate, the error estimate between Tikhonov regularization solution and the exact solution can be derived. According to the optimality theory of regularization, the error estimates are order optimal. Numerical experiment shows that the regularization works well.

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Competing Interests

Authors have declared that no competing interests exist.

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