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Dynamic Buckling Load of an Imperfect Viscously Damped Spherical Cap Stressed by a Step Load

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Authors' contributions

This work was carried out in collaboration between all authors. Author GEO designed the study, and wrote the first draft of the manuscript author EN managed analyses of the study, and author MO managed the literature searches. All authors read and approved the final manuscript.

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ABSTRACT

This paper determines the dynamic buckling load of a lightly and viscously damped imperfect spherical cap with a step load. The spherical cap is discretized into a pre-buckling symmetric mode and a buckling mode that consists of axisymmetric and non-axisymmetric buckling modes. The imperfection is taken at the shape of the buckling mode. The inherent problem contains a small parameter which necessitated the adoption of regular perturbation procedures, using asymptotic technique. The general result is designed to display the contributions of each of the terms in the governing differential equations. We deduce the results for the respective special cases where the axisymmetric imperfection parameter, namely $\bar{\xi}_1$, and the non-axisymmetric imperfection parameter $\bar{\xi}_2$, are zeros. We also determine the effects of each of the non-linear terms as well as the effects of the coupling term.

Keywords: Spherical cap; step load; dynamic buckling; imperfection parameter.

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1. INTRODUCTION

The subject of dynamic buckling of elastic structures has been a thriving area of investigation ever since [1-3] developed the discipline of dynamic stability of elastic structures from the original static consideration that was prevalent before this time. Over the years, many investigations on dynamic stability of elastic structures have been added to the original sketchy and scattered pieces that saw the genesis of dynamic buckling of elastic structures as a research interest. Among the many scholarly investigations that have come to light include [4,5,6,7,8], who investigated the dynamic buckling, of two-degree – of freedom systems with mode interaction under step loading. Mention must also be made of relatively recent investigations which include [9], who investigated the dynamic buckling of thin cylindrical shells under axial impact, [10,11], who studied the nonlinear dynamic buckling of stiffened plates under in-plane impact load.

But by far, the investigation that concerns us in this study is that by [12], who investigated the dynamic buckling loads of imperfection-sensitive structures from perturbation procedures. His analysis was predicated primarily on the studies earlier enunciated by [1-3]. Other Pertinent investigations include those by [13,14,15,16], among others.

However, a cursory appraisal of all the investigations to date reveals that the phenomenon of damping has been given very little or no attention at all in the dynamic buckling process. We are of the strong opinion that since dynamic buckling process is a time dependent process, the effect of damping, no matter how slight, should not be overlooked. In this investigation, the presence of a small viscous damping is therefore assumed and given some level of prominence. Of course, the result obtained is far more representative of the actual physical life situation. To this end, we remark that a few of the many existing investigations that have tended to incorporate damping include the studies by [17-20], among others.

The layout of this investigation is as follows:

We shall first write down the mathematical equations satisfied by the structure investigated.

We shall next develop asymptotic techniques, using perturbation procedures to solve the governing equations analytically. We note that dynamic bucking problems are always non linear and therefore, closed-form exact solutions are not always possible. Therefore, regular perturbation method provides a suitable alternative to the solution of such problems, particularly when the problems contain small parameters in which asymptotic series expansion can always be invoked.

We shall lastly make pertinent deductions.

There are five sections in this paper. Section two examines the dynamic buckling load of an imperfect viscously damped spherical cap stressed by a step load. Section three introduces the viscous damping to Danielson's results. Section four considers the analysis of results while section five ends this work with a conclusion.

2. THE DYNAMIC BUCKLING LOAD

Danielson, had, for simplicity, assumed that the normal displacement $W(x,y,T)$ of the spherical cap was given as

$$W(x,y,T)=\xi_0(T)W_0(x,y)+\xi_1(T)W_1(x,y)+\xi_2(T)W_2(x,y) \tag{1}$$

where $W_0(x,y)$ is the pre-buckling mode and $W_1(x,y),W_2(x,y)$ are the axisymmetric and non-axisymmetiic modes respectively. $\xi_0(T),\xi_1(T)$ and $\xi_2(T)$ are the respective time dependent amplitudes of the associated modes. Imperfection $\bar{W}$ was introduced as

$$\bar{W}=\bar{\xi}_1 W_1+\bar{\xi}_2 W_2 \tag{2}$$

Where W_1, W_2 still have meanings as before and $\bar{\xi}_1, \bar{\xi}_2$ are the imperfect amplitudes assumed to be small relative to unity. On assuming suitable forms for W_0, W_1, W_2 and substituting same into the compatibility and dynamic equilibrium equations and simplifying, using his assumptions, Danielson obtained the following coupled differential equations for step loading.

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2} + \xi_0 = \lambda f(T) \quad (3)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \bar{\xi}_1\,\xi_0 \quad (4)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{dT^2} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2\,\xi_0 \quad (5)$$

$$\xi_i(0) = \xi_i^{/}(0) = 0; i = 1, 2.$$

Here, $f(T)$ is the loading history which in our investigation, (as in Danielson's case), is the step load characterized by

$$f(T) = \begin{cases} 1, T>0 \\ 0, T<0 \end{cases} , \quad (6)$$

and, λ ,is the load parameter, considered to be non-dimensionalized and satisfies the inequality $0 < \lambda < 1$.

In our guest for solution, we are to determine a particular value of λ , called the dynamic buckling load represented by λ_D and which satisfies the inequality $0 < \lambda_D < 1$. We define the dynamic buckling load λ_D as the largest load parameter such that the solution to the damped version of problems (3)-(6) remains bounded for all time T>0. As in (3)-(5), we note that $\omega_i; i = 0, 1, 2$ are the circular frequencies of the associated modes ξ_0, ξ_1 and ξ_2 respectively while k_2 and k_2 are constants considered positive.

3. THE USE OF VISCOUS DAMPING IN DANIELSON'S RESULTS

The present study is an extension of Danielson's problem to the case where a small viscous damping is present. We however avoid Danielson's method (who used Mathieu – type of instability), for, as noted by [3, page 100], Mathieu – type of instability is always associated with many cycles of oscillations as opposed to just one shot of oscillation that triggers off dynamic buckling.

For simplicity of analysis, we assume the existence of damping on the buckling modes. Since this damping must be only proportional to the velocity, we add the terms $c_1\frac{d\xi_1}{dT}$ and $c_2\frac{d\xi_2}{dT}$ to (4) and (5) respectively and the formulation now becomes

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2} + \xi_0 = \lambda f(T) \quad (7)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2} + c_1\frac{d\xi_1}{dT} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \bar{\xi}_1\,\xi_0 \quad (8)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{dT^2} + c_2\frac{d\xi_2}{dT} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2\,\xi_0 \quad (9)$$

where $c_i, i = 1, 2$ are the damping constants and which satisfy the inequality $0 < c_i < 1$.

Using $f(T) = 1$ and substituting (6) into (7) we have

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2} + \xi_0 = \lambda \quad (10)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2} + c_1\frac{d\xi_1}{dT} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \bar{\xi}_1\,\xi_0 \quad (11)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{dT^2} + c_2\frac{d\xi_2}{dT} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2\,\xi_0 \quad (12)$$

Now using,

$$t = \omega_0 T,$$

so that

$$\frac{d(\ )}{dT} = \omega_0 \frac{d(\ )}{dt}, \frac{d^2(\ )}{dT^2} = \omega_o^2 \frac{d^2(\ )}{dt^2},$$

Then $(10)-(12)$ become

$$\frac{d^2\xi_0}{dt^2} + \xi_0 = \lambda \tag{13}$$

$$\frac{d^2\xi_1}{dt^2} + \left[\frac{c_1\omega_0\omega_1^2}{\omega_0^2}\right]\frac{d\xi_1}{dt} + \left[\frac{\omega_1}{\omega_0}\right]^2 \xi_1(1-\xi_0) - \left[\frac{\omega_1}{\omega_0}\right]^2 k_1\xi_1^2 + \left[\frac{\omega_1}{\omega_0}\right]^2 k_2\xi_2^2 = \left[\frac{\omega_1}{\omega_0}\right]^2 \bar{\xi}_1 \xi_0 \tag{14}$$

$$\frac{d^2\xi_2}{dt^2} + \left[\frac{c_2\omega_0\omega_2^2}{\omega_0^2}\right]\frac{d\xi_2}{dt} + \left[\frac{\omega_2}{\omega_0}\right]^2 \xi_2(1-\xi_0) + \left[\frac{\omega_2}{\omega_0}\right]^2 \xi_1\xi_2 = \left[\frac{\omega_2}{\omega_0}\right]^2 \bar{\xi}_2 \xi_0 \tag{15}$$

Next, we let

$$2\alpha_1\varepsilon = \frac{c_1\omega_1^2}{\omega_0}, 2\alpha_2\varepsilon = \frac{c_2\omega_2^2}{\omega_0}, Q = \frac{\omega_1}{\omega_0}, R = \frac{\omega_2}{\omega_0}, S = \left[\frac{\omega_2}{\omega_1}\right]^2 \tag{16}$$

where,

$$\varepsilon = \lambda Q^2 = \lambda\left[\frac{\omega_1}{\omega_0}\right]^2, \tag{17}$$

and

$$0 < \alpha_1 < 1,\ 0 < \alpha_2 < 1,\ 0 < Q < 1,\ 0 < R < 1 \, and \ 0 < \varepsilon < 1$$

Substituting (16) into (14) and (15) yield

$$\frac{d^2\xi_0}{dt^2} + \xi_0 = \lambda \tag{18}$$

$$\frac{d^2\xi_1}{dt^2} + 2\alpha_1\varepsilon\frac{d\xi_1}{dt} + Q^2\xi_1(1-\xi_0) - k_1Q^2\xi_1^2 + k_2Q^2\xi_2^2 = Q^2 \bar{\xi}_1 \xi_0 \tag{19}$$

$$\frac{d^2\xi_2}{dt^2} + 2\alpha_2\varepsilon\frac{d\xi_2}{dt} + R^2\xi_2(1-\xi_0) + R^2\xi_1\xi_2 = R^2\bar{\xi}_2\xi_0 \tag{20}$$

$$\xi_i(0) = \xi_i^{/}(0) = o;\ i = 1, 2.$$

As in $[1-3]$, we neglect the pre-buckling inertia term, so that from (18) we get

$$\xi_0 = \lambda \tag{21}$$

Here, we assume zero pre-buckling inertia term since the load imparts zero initial displacement and velocity to the pre-buckling mode.

On simplification, using (21), equations (19) and (20) yield

$$\frac{d^2\xi_1}{dt^2} + 2\alpha_1\varepsilon\frac{d\xi_1}{dt} + Q^2\xi_1 - \varepsilon\xi_1 - k_1Q^2\xi_1^2 + k_2Q^2\xi_2^2 = \varepsilon\bar{\xi}_1 \tag{22}$$

and

$$\frac{d^2\xi_2}{dt^2} + 2\alpha_2\varepsilon\frac{d\xi_2}{dt} + R^2\xi_2 - \varepsilon S\xi_2 + R^2\xi_1\xi_2 = \varepsilon S\bar{\xi}_2 \tag{23}$$

$$\xi_i(0) = \xi_i^{/}(0) = 0;\ i = 1, 2$$

where,

$$S = \left[\frac{R}{Q}\right]^2.$$

We assume a small time scale τ such that,

$$\tau = \varepsilon t \tag{24a}$$

and

$$\xi_i^{/} = \xi_{i,t} + \varepsilon\xi_{i,\tau} \tag{24b}$$

$$\xi_i^{//} = \xi_{i,tt} + 2\varepsilon\xi_{i,t\tau} + \varepsilon^2\xi_{i,\tau\tau};\ i = 1,2 \tag{24c}$$

We denote our perturbation parameter by ε so that

$$\xi_1(t) = \sum_{i=1}^{\infty}\varsigma^{(i)}(t,\tau)\varepsilon^i \tag{25}$$

$$\xi_2(t) = \sum_{i=1}^{\infty}\eta^{(i)}(t,\tau)\varepsilon^i \tag{26}$$

Substituting (25) and (26) into (22) and (23), using (24b) and (24c), equating terms of the orders of ε we get,

$$\varsigma_{,tt}^{(1)} + Q^2\varsigma^{(1)} = \bar{\xi}_1 \tag{27}$$

$$\varsigma_{,tt}^{(2)} + Q^2\varsigma^{(2)} = -2\alpha_1\varsigma_{,t}^{(1)} + \varsigma^{(1)} + k_1Q^2\varsigma^{(1)^2} - k_2Q^2\eta^{(1)^2} - 2\varsigma_{,t\tau}^{(1)} \tag{28}$$

and

$$\eta_{,tt}^{(1)} + R^2\eta^{(1)} = S\bar{\xi}_2 \tag{29}$$

$$\eta_{,tt}^{(2)} + R^2\eta^{(2)} = -2\alpha_2\eta_{,t}^{(1)} + S\eta^{(1)} - 2\eta_{,t\tau}^{(1)} - R^2\varsigma^{(1)}\eta^{(1)} \tag{30}$$

$$\varsigma^{(i)}(0,0) = \eta^{(i)}(0,0) = 0, i = 1,2 \tag{31}$$

$$\varsigma_{,t}^{(i)}(0,0) = \eta_{,t}^{(i)}(0,0) = 0, i = 1,2 \tag{32}$$

$$\varsigma_{,t}^{(i+1)}(0,0) + \varsigma_{,\tau}^{(i)}(0,0) = \eta_{,t}^{(i+1)}(0,0) + \eta_{,\tau}^{(i)}(0,0) = 0, i = 1,2 \tag{33}$$

The solution of (27) using (31) and (32) is

$$\varsigma^{(1)}(t,\tau) = a_1(\tau)\cos Qt + b_1(\tau)\sin Qt + \frac{\bar{\xi}_1}{Q^2} \tag{34a}$$

$$a_1(0) = -\frac{\bar{\xi}_1}{Q^2}; b_1(0) = 0 \tag{34b}$$

Similarly, the solution of (28) is

$$\eta^{(1)}(t,\tau) = a_2(\tau)\cos Rt + b_2(\tau)\sin Rt + \frac{S\bar{\xi}_2}{R^2} \tag{35a}$$

$$a_2(0) = -\frac{S\bar{\xi}_2}{R^2}; b_2(0) = 0 \tag{35b}$$

Substituting using (34a) and (35a) into (28), we have

$$\varsigma_{,tt}^{(2)} + Q^2\varsigma^{(2)} = -2\alpha_1\left[-Qa_1\sin Q\mathrm{t} + Q\mathrm{b}_1\cos Q\mathrm{t}\right] + \left[a_1\cos Q\mathrm{t} + \mathrm{b}_1\sin Q\mathrm{t} + \frac{\bar{\xi}_1}{Q^2}\right]$$

$$-k_2 Q^2\left[\frac{1}{2}[a_2^2+b_2^2]+a_2 b_2 \sin 2\mathrm{Rt}+\frac{1}{2}[a_2^2-b_2^2]\cos 2\mathrm{Rt}\right]$$

$$-k_2 Q^2\left[+\frac{2a_2\,\mathrm{S}\bar{\xi}_2}{\mathrm{R}^2}\cos\mathrm{Rt}+\frac{2b_2\,\mathrm{S}\bar{\xi}_2}{\mathrm{R}^2}\sin\mathrm{Rt}\right]$$

$$+k_1 Q^2\left[\frac{1}{2}[a_1^2+b_1^2]+\mathrm{a}_1\mathrm{b}_1\sin 2Q\mathrm{t}+\frac{1}{2}[a_1^2-b_1^2]\cos 2Q\mathrm{t}\right]$$

$$+k_1 Q^2\left[+\frac{2\,\mathrm{a}_1\bar{\xi}_1}{Q^2}\cos Q\,\mathrm{t}+\frac{2\,\mathrm{b}_1\bar{\xi}_1}{Q^2}\sin Q\,\mathrm{t}\right]+2Q\left[\mathrm{a}_1'\sin Q\,\mathrm{t}-\mathrm{b}_1'\cos Q\,\mathrm{t}\right] \quad (36)$$

To maintain a uniformly valid asymptotic solution in time scale t, we equate the coefficients of $\cos Qt$ and $\sin Qt$ to zero to get (on the rhs of 36). This ensures a finite at infinite time, i.e. as t tends to infinity, such terms tends to zero. The terms are called secular terms, we therefore device a way of eliminating the terms, hence making the solution bounded for all t

$$\cos Qt :- \; -2\alpha_1 b_1 Q + a_1 + \frac{2a_1 k_1 \bar{\xi}_1 Q^2}{Q^2} - 2b_1' Q = 0$$

On simplification, we get

$$b_1' + \alpha_1 b_1 = \frac{a_1}{2Q}\left[1+2k_1\bar{\xi}_1\right]$$

$$b_1' + \alpha_1 b_1 = a_1\varphi \quad (37a)$$

Similarly,

$$\sin Qt \; :- \; 2\alpha_1 a_1 Q + b_1 + \frac{2b_1 k_1 \bar{\xi}_1 Q^2}{Q^2} + 2a_1' Q = 0$$

Simplifying gives

$$a_1' + \alpha_1 a_1 = -\frac{b_1}{2Q}\left[1+2k_1\bar{\xi}_1\right]$$

$$a_1' + \alpha_1 a_1 = -b_1\varphi \quad (37b)$$

where,

$$(\;)' = \frac{d(\;)}{d\tau},$$

and

$$\varphi = \frac{1}{2Q}\left[1 + 2k_1 \bar{\xi}_1\right]$$

Simplification of (37a, b) yield

$$b_1^{//} + \alpha_1 b_1^{/} = -\varphi\left[b_1\left[\varphi + \frac{\alpha_1^2}{\varphi}\right] + \frac{\alpha_1 b_1^{/}}{\varphi}\right]$$

$$b_1^{//} + 2\alpha_1 b_1^{/} + \varphi b_1\left[\varphi + \frac{\alpha_1^2}{\varphi}\right] = 0$$

$$b_1(0) = 0; b_1^{/}(0) = -\frac{\varphi \bar{\xi}_1}{Q^2} \tag{37c}$$

and

$$a_1^{//} + \alpha_1 a_1^{/} = -\varphi\left[a_1\left[\varphi + \frac{\alpha_1^2}{\varphi}\right] + \frac{\alpha_1 a_1^{/}}{\varphi}\right]$$

$$a_1^{//} + 2\alpha_1 a_1^{/} + \varphi a_1\left[\varphi + \frac{\alpha_1^2}{\varphi}\right] = 0$$

$$a_1(0) = -\frac{\bar{\xi}_1}{Q^2}; a_1^{/}(0) = \frac{\alpha_1 \bar{\xi}_1}{Q^2} \tag{37d}$$

The remaining part of the equation in the substitution into (28) as obtained from (36) is

$$\varsigma_{,tt}^{(2)} + Q^2\varsigma^{(2)} = q_1 + k_1 Q^2\left[p_0(\tau)\sin 2Qt + p_1(\tau)\cos 2Qt\right]$$
$$- k_2 Q^2\left[p_2(\tau)\sin 2Rt + p_3(\tau)\cos 2Rt + p_4(\tau)\cos Rt + p_5(\tau)\sin Rt\right] \tag{38a}$$

$$\varsigma^{(2)}(0,0) = 0; \varsigma_{,t}^{(2)}(0,0) + \varsigma_{,\tau}^{(2)}(0,0) = 0 \tag{38b}$$

where,

$$q_1 = \frac{\bar{\xi}_1}{Q^2} + k_1 Q^2 r_0(\tau) - k_2 Q^2 r_1(\tau); p_0(\tau) = a_1 b_1; p_1(\tau) = \frac{1}{2}\left[a_1^2 - b_1^2\right] \tag{38c}$$

$$p_2(\tau) = a_2 b_2; p_3(\tau) = \frac{1}{2}\left[a_2^2 - b_2^2\right]; p_4(\tau) = \frac{2a_2 S \bar{\xi}_2}{R^2}; p_5(\tau) = \frac{2b_2 S \bar{\xi}_2}{R^2} \tag{38d}$$

$$r_0(\tau) = \frac{1}{2}\left[a_2^2 + b_2^2\right]; r_1(\tau) = \frac{1}{2}\left[a_1^2 + b_1^2\right] \tag{38e}$$

$$p_0(0)=0; p_1(0)=\frac{\bar{\xi}_1^2}{2Q^4}; p_2(0)=0; p_3(0)=\frac{S^2\bar{\xi}_2^2}{2R^4} \tag{38f}$$

$$p_4(0)=\frac{2S^2\bar{\xi}_2^2}{R^4}; p_5(0)=0; r_0(0)=\frac{\bar{\xi}_1^2}{2Q^4}; r_1(0)=\frac{S^2\bar{\xi}_2^2}{2R^4}; \tag{38g}$$

The solution of (38a), using (38b) is

$$\varsigma^{(2)}(t,\tau)=a_3(\tau)\cos Qt+b_3(\tau)\sin Qt+\frac{q_1}{Q^2}-\frac{k_1}{3}\left[p_6(\tau)\sin 2Qt+p_7(\tau)\cos 2Qt\right]$$

$$-k_2Q^2\left[p_8(\tau)\sin 2Rt+p_9(\tau)\cos 2Rt+p_{10}(\tau)\cos Rt+p_{11}(\tau)\sin Rt\right] \tag{39a}$$

$$a_3(0)=\bar{\xi}_1 l_0+k_1\bar{\xi}_1^2 l_1+k_2\bar{\xi}_2^2\left[\frac{S}{R^2}\right]^2 l_2; b_3(0)=-\frac{\alpha_1\bar{\xi}_1}{Q^3} \tag{39b}$$

where

$$l_0=-\frac{1}{Q^4}; l_1=-\frac{1}{2Q^2}+\frac{1}{6Q^4}; l_2=\frac{1}{2}+\frac{1}{2\left[Q^2-4R^2\right]}-\frac{2}{R^2\left[Q^2-4R^2\right]} \tag{39c}$$

$$p_6(\tau)=a_1b_1; p_7(\tau)=a_2b_2; p_8(\tau)=\frac{p_2(\tau)}{Q^2-4R^2}; p_9(\tau)=\frac{p_3(\tau)}{Q^2-4R^2} \tag{39d}$$

$$p_{10}(\tau)=\frac{p_4(\tau)}{Q^2-R^2}; p_{11}(\tau)=\frac{p_5(\tau)}{Q^2-R^2}; p_{11}(0)=0 \tag{39e}$$

$$p_6(0)=0; p_7(0)=\frac{\bar{\xi}_1^2}{2Q^4}; p_8(0)=0; p_9(0)=\frac{S^2\xi_2^2}{2R^4\left[Q^2-4R^2\right]}; p_{10}(0)=\frac{2S^2\bar{\xi}_2^2}{R^4\left[Q^2-R^2\right]} \tag{39f}$$

Substituting using (34a) and (35a) into (30) we get,

$$\eta_{,tt}^{(2)}+R^2\eta^{(2)}=-2\alpha_2\left[-Ra_2\sin Rt+Rb_2\cos Rt\right]-2R\left[-a_2'\sin Rt+b_2'\cos Rt\right]+S$$
$$\left[a_2\cos Rt+b_2\sin Rt+\frac{S\xi_2}{R^2}\right]-\frac{R^2}{2}$$

$$\begin{bmatrix} \frac{2S\bar{\xi}_1\bar{\xi}_2}{[QR]^2} + \frac{2a_2\bar{\xi}_1}{Q^2}\cos Rt + \frac{2b_2\bar{\xi}_1}{Q^2}\sin Rt + \frac{2a_1S\bar{\xi}_2}{R^2}\cos Q\,t + \frac{2b_1S\bar{\xi}_2}{R^2}\sin Q\,t \\ +[a_1a_2 - b_1b_2]\cos[Q-R]t + [a_1b_2 + b_1a_2]\sin[Q+R]t \\ +[a_1a_2 + b_1b_2]\cos[Q-R]t + [b_1a_2 - a_1b_2]\sin[Q-R]t \end{bmatrix} \quad (40)$$

Now, to ensure a uniformly valid asymptotic solution in tine scale t, we equate the coefficients $\cos Rt$ and $\sin Rt$ to zero. This will ensure a finite at infinite time, i.e. as t tends to infinity, such terms tends to zero thereby making the solution not to be bounded, hence non-uniform. Such terms are called secular terms and our aim is to get rid of them.

$$\cos Rt :- \; -2\alpha_2 b_2 R - 2b_2^{/} R + S\alpha_2 - \frac{a_2\bar{\xi}_1 R^2}{Q^2} = 0$$

Implies that,

$$b_2^{/} + \alpha_2 b_2 = \frac{a_2}{2R}\left[S - \frac{\bar{\xi}_1 R^2}{Q^2}\right]$$

$$b_2^{/} + \alpha_2 b_2 = a_2\Phi \quad (41a)$$

Similarly,

$$\sin Rt :- \; 2\alpha_2 a_2 R + 2a_2^{/} R + Sb_2 - \frac{b_2\bar{\xi}_1 R^2}{Q^2} = 0$$

Simplifying gives

$$a_2^{/} + \alpha_2 a_2 = -\frac{b_2}{2R}\left[S - \frac{\bar{\xi} R^2}{Q^2}\right]$$

$$a_2^{/} + \alpha_2 a_2 = -b_2\Phi \quad (41b)$$

where,

$$\Phi = \frac{1}{2R}\left[S - \frac{\bar{\xi}_1 R^2}{Q^2}\right].$$

Simplification of (41a, b) yield

$$b_2^{//} + \alpha_2 b_2^{/} = -\Phi\left[\Phi b_2 + \alpha_2 a_2\right]$$

$$b_2^{//} + \alpha_2 b_2^{/} = -\Phi\left[\Phi b_2 + \frac{\alpha_2}{\Phi}\left[b_2^{/} + \alpha_2 b_2\right]\right]$$

$$b_2^{//} + 2\alpha_2 b_2^{/} + b_2\left[\Phi^2 + \alpha_2^2\right] = 0$$

$$b_2(0) = 0; b_2^{/}(0) = -\frac{\Phi S\bar{\xi}_2}{R^2} \quad (41c)$$

and

$$a_2^{//} + \alpha_2 a_2^{/} = -\Phi\left[\Phi a_2 - \alpha_2 b_2\right]$$

$$a_2^{//} + \alpha_2 a_2^{/} = -\Phi\left[\Phi a_2 + \frac{\alpha_2}{\Phi}\left[a_2^{/} + \alpha_2 a_2\right]\right]$$

$$a_2^{//} + 2\alpha_2 a_2^{/} + a_2\left[\Phi^2 + \alpha_2^2\right] = 0$$

$$a_2(0) = -\frac{S\bar{\xi}_2}{R^2}; a_2^{/}(0) = -\frac{\alpha_2 S\bar{\xi}_2}{R^2} \quad (41d)$$

The remaining part of the equation in the substitution into (30) as obtained from (40) is

$$\eta^{(2)}_{,tt} + R^2\eta^{(2)} = q_2 - \frac{R^2}{2}\begin{bmatrix} p_{12}(\tau)\cos Qt + p_{13}(\tau)\sin Qt + p_{14}(\tau)\cos[Q+R]t \\ + p_{15}(\tau)\sin[Q+R]t + p_{16}(\tau)\cos[Q-R]t + p_{17}(\tau)\sin[Q-R] \end{bmatrix} \quad (42a)$$

$$\eta^{(2)}(0,0) = 0; \eta^{(2)}_{,t}(0,0) + \eta^{(1)}_{,\tau}(0,0) = 0 \quad (42b)$$

where,

$$q_2 = \frac{S^2\bar{\xi}_2}{R^2} - \frac{S\bar{\xi}_1\bar{\xi}_2}{Q^2}; p_{12}(\tau) = \frac{2a_1S\bar{\xi}_2}{R^2}; p_{13}(\tau) = \frac{2b_1S\bar{\xi}_2}{R^2}; p_{14}(\tau) = a_1a_2 - b_1b_2 \quad (42c)$$

$$p_{15}(\tau) = b_1b_2 + b_1a_2; p_{16}(\tau) = a_1a_2 + b_1b_2; p_{17}(\tau) = b_1a_2 - a_1b_1 \quad (42d)$$

$$p_{12}(0) = \frac{2S\bar{\xi}_1\bar{\xi}_2}{Q^2R^2}; p_{13}(0) = 0; p_{14}(0) = \frac{S\bar{\xi}_1\bar{\xi}_2}{Q^2R^2}; p_{15}(0) = 0; \quad (42e)$$

$$p_{16}(0) = \frac{S\bar{\xi}_1\bar{\xi}_2}{Q^2R^2}; p_{17}(0) = 0 \quad (42f)$$

The solution of (42a) using (42b) is

$$\eta^{(2)}(t,\tau) = a_4(\tau)\cos Rt + b_4(\tau)\sin Rt + \frac{q_2}{R^2} - \frac{1}{2}\begin{bmatrix} p_{18}(\tau)\cos Qt + p_{19}(\tau)\sin Qt + \\ p_{20}(\tau)\cos[Q+R]t + p_{21}(\tau)\sin[Q+R]t + \\ p_{22}(\tau)\cos[Q-R]t + p_{23}(\tau)\sin[Q-R]t \end{bmatrix} \quad (43a)$$

$$a_4(0) = S^2\bar{\xi}_2 l_3 + R^2S\bar{\xi}_1\bar{\xi}_2 l_4; b_4(0) = -\frac{\alpha_2S\bar{\xi}_2}{R^3} \quad (43b)$$

where,

$$l_4 = \left[\frac{1}{[R^2Q]^2} + \frac{1}{2}\left[\frac{-2}{[RQ]^2[R^2-Q^2]} - \frac{1}{Q[RQ]^2[2R+Q]} + \frac{1}{Q[RQ]^2[2R-Q]}\right]\right] \quad (43c)$$

$$l_3 = -\frac{1}{R^4}; p_{18}(\tau) = \frac{p_{12}(\tau)}{R^2-Q^2}; p_{19}(\tau) = \frac{p_{13}(\tau)}{R^2-Q^2}; p_{20}(\tau) = \frac{p_{14}(\tau)}{Q[2R+Q]} \quad (43d)$$

$$p_{21}(\tau) = \frac{p_{15}(\tau)}{Q[2R+Q]}; p_{22}(\tau) = \frac{p_{16}(\tau)}{Q[2R-Q]}; p_{23}(\tau) = \frac{p_{17}(\tau)}{Q[2R-Q]} \quad (43e)$$

$$p_{18}(0) = \frac{2S\bar{\xi}_1\bar{\xi}_2}{Q^2R^2[R^2-Q^2]}; p_{19}(0) = 0; p_{20}(0) = \frac{S\bar{\xi}_1\bar{\xi}_2}{Q^3R^2[2R+Q]} \quad (43f)$$

$$p_{21}(0)=0; p_{22}(0)=\frac{S\bar{\xi}_1\bar{\xi}_2}{Q^2R^2[2R-Q]}; p_{23}(0)=0 \tag{43g}$$

Next, using (34a), (39a) and (35a), (43a) we deduce the displacements as

$$\xi_1(t)=\varsigma^{(1)}(t,\tau)\varepsilon+\varsigma^{(2)}(t,\tau)\varepsilon^2+... \tag{44a}$$

and

$$\xi_2(t)=\eta^{(1)}(t,\tau)\varepsilon+\eta^{(2)}(t,\tau)\varepsilon^2+... \tag{44b}$$

We seek the maximum displacement for both $\xi_1(t)$ and $\xi_2(t)$. To achieve this, we shall first determine the critical values of t and τ for each of $\xi_1(t)$ and $\xi_2(t)$ at their maximum values. The condition for the maximum displacements of $\xi_1(t)$ and $\xi_2(t)$ is obtain from (24b). hence

$$\xi_{1,t}+\varepsilon\xi_{1,\tau}, \tag{45a}$$

$$\xi_{2,t}+\varepsilon\xi_{2,\tau}, \tag{45b}$$

We know from (44a, b) that

$$\xi_1(t)=\varsigma^{(1)}(t,\tau)\varepsilon+\varsigma^{(2)}(t,\tau)\varepsilon^2+... \tag{46a}$$

$$\xi_2(t)=\eta^{(1)}(t,\tau)\varepsilon+\eta^{(2)}(t,\tau)\varepsilon^2+... \tag{46b}$$

On applying (45a, b) to (46a, b), we get

$$\varsigma_{,t}+\varepsilon\varsigma_{,\tau}=\left[\varsigma_{,t}^{(1)}(t_a,\tau_a)\varepsilon+\varsigma_{,t}^{(2)}(t_a,\tau_a)\varepsilon^2+...\right]$$
$$+\varepsilon\left[\varsigma_{,\tau}^{(1)}(t_a,\tau_a)\varepsilon+\varsigma_{,\tau}^{(2)}(t_a,\tau_a)\varepsilon^2+...\right]=0 \tag{47a}$$

and

$$\eta_{,t}+\varepsilon\eta_{,\tau}=\left[\eta_{,t}^{(1)}(T_c,\tau_c)\varepsilon+\eta_{,t}^{(2)}(T_c,\tau_c)\varepsilon^2+...\right]$$
$$+\varepsilon\left[\eta_{,\tau}^{(1)}(T_c,\tau_c)\varepsilon+\eta_{,\tau}^{(2)}(T_c,\tau_c)\varepsilon^2+...\right]=0 \tag{47b}$$

where, (t_a,τ_a) and (T_c,τ_c) are the values of t and τ at the maximum displacement of $\varsigma(t,\tau)$ and $\eta(t,\tau)$ respectively.

We now expand (47a, b) in a Taylor series about $t_a=t_0,\tau_a=0$ and $T_c=T_0,\tau_c=0$, and thereafter equate to zero the terms of the same orders of ε to get

$$\varsigma_{,t}^{(1)}(t_0,0)=0 \tag{48a}$$

$$t_1\varsigma_{,tt}^{(1)}(t_0,0)+t_0\varsigma_{,t\tau}^{(1)}(t_0,0)+\varsigma_{,t}^{(2)}(t_0,0)+\varsigma_{,\tau}^{(1)}(t_0,0)=0 \tag{48b}$$

and

$$\eta_{,t}^{(1)}(T_0,0)=0 \tag{49a}$$

$$T_1\eta_{,tt}^{(1)}(T_0,0)+T_0\eta_{,t\tau}^{(1)}(T_0,0)+\eta_{,t}^{(2)}(T_0,0)+\eta_{,\tau}^{(1)}(T_0,0)=0 \tag{49b}$$

Substituting for $\varsigma_{,t}^{(1)}$ from (34a) in (48a) and simplifying we get

$$\sin Qt_0=0 \tag{50a}$$

A further simplification of (50a) gives

$$t_0=\frac{\pi}{Q} \tag{50b}$$

A similar solution for (49a) is

$$T_0=\frac{\pi}{R} \tag{50c}$$

Next, we deduce from (48b) that

$$t_1=-\frac{1}{\varsigma_{,tt}^{(1)}(t_0,0)}\left[t_0\varsigma_{,t\tau}^{(1)}(t_0,0)+\varsigma_{,t}^{(2)}(t_0,0)+\varsigma_{,\tau}^{(1)}(t_0,0)\right] \tag{51a}$$

Simplification of the following terms are however necessary in this analysis,

$$\varsigma_{,t}^{(2)}(t_0,0)=\alpha_1\bar{\xi}_1 l_5+k_1\bar{\xi}_1^2 l_6-k_2S^2\bar{\xi}_2^2 l_7;\varsigma_{,t}^{(1)}(t_0,0)=\bar{\xi}_1 l_8 \tag{51b}$$

$$\varsigma_{,\tau}^{(1)}(t_0,0)=\bar{\xi}_1 l_9;\varsigma_{,tt}^{(1)}(t_0,0)=\bar{\xi}_1;\varsigma^{(1)}(t_0,0)=2\bar{\xi}_1 l_{10} \tag{51c}$$

$$\varsigma^{(2)}(t_0,0)=2\bar{\xi}_1 l_{11}+k_1\bar{\xi}_1^2 l_{12}+k_2\bar{\xi}_2^2\left[\frac{S}{R^2}\right]^2 l_{13};\varsigma_{,t}^{(1)}(t_0,0)=0; \tag{51d}$$

where

$$l_5=\frac{1}{Q^2};l_6=\frac{Sin2Qt_0}{3Q^2};l_7=\frac{Q^2Sin2Rt_0}{R^3\left[Q^2-4R^2\right]} \tag{51e}$$

$$l_8=\frac{\varphi}{Q};l_9=-\frac{\alpha_1}{Q^2};l_{10}=\frac{1}{Q^2};l_{11}=\frac{1}{Q^4};l_{12}=\frac{1}{Q^2}-\frac{1}{3Q^4} \tag{51f}$$

$$l_{13} = \left[-1 - \frac{1}{2\left[Q^2 - 4R^2\right]} + \frac{1}{R^2\left[Q^2 - 4R^2\right]} - Q^2 \left[\frac{1}{Q^2 - 4R^2} + \frac{2}{Q^2 - R^2} \right] \right] \tag{51g}$$

On substituting (51, b-d) on (51a), we have

$$t_1 = \alpha_1 l_5 + k_1 \bar{\xi}_1 l_6 - k_2 S \bar{\xi}_2 l_7 + t_0 l_8 + l_9 \tag{52}$$

Similarly, deducing from (49b) yields

$$T_1 = -\frac{1}{\eta^{(1)}_{,tt}(T_0,0)} \left[T_0 \eta^{(1)}_{,t\tau}(T_0,0) + \eta^{(2)}_{,t}(T_0,0) + \eta^{(1)}_{,\tau}(t_0,0) \right] \tag{53a}$$

We however note the following simplifications

$$\eta^{(2)}_{,t}(T_0,0) = \alpha_2 S \bar{\xi}_2 l_{14} + S^2 \bar{\xi}_2 l_{15} + S \bar{\xi}_1 \bar{\xi}_2 l_{16} ; \eta^{(1)}_{,t\tau}(T_0,0) = S \bar{\xi}_2 l_{17} \tag{53b}$$

$$\eta^{(1)}_{,\tau}(T_0,0) = S^2 \bar{\xi}_2 l_{18} ; \eta^{(1)}_{,tt}(T_0,0) = -S \bar{\xi}_2 ; \eta^{(1)}(T_0,0) = 2S \bar{\xi}_2 l_{20} \tag{53c}$$

$$\eta^{(2)}(T_0,0) = S^2 \bar{\xi}_2 l_{21} + R^2 S \bar{\xi}_1 \bar{\xi}_2 l_{19} ; \eta^{(1)}_{,t}(T_0,0) = 0; \tag{53d}$$

where

$$l_{14} = -\frac{\cos RT_0}{R^2} ; l_{15} = -R l_3 \sin RT_0 \tag{53e}$$

$$l_{16} = -R^3 S l_4 \sin RT_0 - \frac{R^2}{2} \left[\begin{array}{l} \dfrac{2\sin QT_0}{QR^2\left[R^2 - Q^2\right]} - \dfrac{\cos QT_0}{[RQ]^2[2R+Q]} \\ - \dfrac{[Q+R]\sin[Q+R]T_0}{Q[RQ]^2[2R+Q]} - \dfrac{[Q-R]\sin[Q-R]T_0}{Q[RQ]^2[2R-Q]} \end{array} \right] \tag{53f}$$

$$l_{17} = \frac{\Phi}{R} ; l_{18} = -\frac{\alpha_2}{R^2} ; l_{20} = \frac{1}{R^2} ; l_{21} = \frac{1}{R^4} ; l_{21} = \frac{1}{R^4} \tag{53g}$$

$$l_{19} = \left[-l_4 + \frac{1}{2} \left[\frac{2\cos QT_0}{Q^2 R^2 \left[R^2 - Q^2\right]} + \frac{\cos[Q+R]T_0}{Q[RQ]^2[2R+Q]} + \frac{\cos[Q-R]T_0}{Q[RQ]^2[2R-Q]} \right] \right] \tag{53h}$$

On substituting (53, b-d) on (53a), we have

$$T_1 = \alpha_2 l_{14} + \bar{\xi}_1 l_{16} + T_0 l_{17} + l_{18} \tag{54}$$

We, now, determine the maximum values of $\varsigma(t)$ and $\eta(t)$ say ς_a and η_c respectively by evaluating (46 a, b) at the critical values namely $t = t_a, \tau = \tau_a$ and $T = T_c, \tau = \tau_c$.

$$\varsigma_a = \varsigma^{(1)}(t_a, \tau_a)\varepsilon + \varsigma^{(2)}(t_a, \tau_a)\varepsilon^2 + \ldots \tag{55a}$$

$$\eta_c = \eta^{(1)}(T_c, \tau_c)\varepsilon + \eta^{(2)}(T_c, \tau_c)\varepsilon^2 + \ldots \tag{55b}$$

Expanding (55 a) in Taylor series using,

$$t_a = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \ldots; \tau_a = \varepsilon t_a = \varepsilon\left[t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \ldots\right] \tag{56a}$$

we have

$$\begin{aligned}\varsigma_a = \varepsilon\Big[&\varsigma^{(1)}(t_0,0) + \varsigma^{(1)}_{,t}(t_0,0)\left[\varepsilon t_1 + \varepsilon^2 t_2 + \ldots\right] + \varsigma^{(1)}_{,\tau}(t_0,0)\varepsilon\left[t_0 + t_1\varepsilon_1 + \ldots\right]\Big] \\ &+ \varsigma^{(2)}(t_0,0)\varepsilon^2 + \ldots\end{aligned} \tag{56b}$$

Regrouping the terms in orders of ε yields

$$\varsigma_a = \varepsilon\varsigma^{(1)}(t_0,0) + \varepsilon^2\left[t_1\varsigma^{(1)}_{,t}(t_0,0) + t_0\varsigma^{(1)}_{,\tau}(t_0,0) + \varsigma^{(2)}(t_0,0)\right] + \ldots \tag{56c}$$

On substituting the terms in (56c) from (51, b-d), we have

$$\varsigma_a = 2\bar{\xi}_1 l_{10}\varepsilon + \left[t_0\bar{\xi}_1 l_9 + 2\bar{\xi}_1 l_{11} + k_1\bar{\xi}_1^2 l_{12} + k_2\bar{\xi}_2^2\left[\frac{S}{R^2}\right]^2 l_{13}\right]\varepsilon^2 + \ldots \tag{57}$$

Similarly, expanding (55 b) in Taylor series using,

$$T_c = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots; \tau_c = \varepsilon T_c = \varepsilon\left[T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots\right] \tag{58a}$$

we have

$$\begin{aligned}\eta_c = \varepsilon\Big[&\eta^{(1)}(T_0,0) + \eta^{(1)}_{,t}(T_0,0)\left[\varepsilon T_1 + \varepsilon^2 T_2 + \ldots\right] + \eta^{(1)}_{,\tau}(T_0,0)\varepsilon\left[T_0 + \varepsilon_1 T_1 + \ldots\right]\Big] \\ &+ \eta^{(2)}(T_0,0)\varepsilon^2 + \ldots\end{aligned} \tag{58b}$$

Regrouping the terms in orders of ε yields

$$\eta_c = \varepsilon\eta^{(1)}(T_0,0) + \varepsilon^2\left[T_1\eta^{(1)}_{,t}(T_0,0) + T_0\eta^{(1)}_{,\tau}(T_0,0) + \eta^{(2)}(T_0,0)\right] + \ldots \tag{58c}$$

On substituting the terms in (58c) from (53, b-d), we have

$$\eta_c = 2S\bar{\xi}_2 l_{20}\varepsilon + \left[T_0 S^2\bar{\xi}_2 l_{18} + S^2\bar{\xi}_2 l_{21} + R^2 S\bar{\xi}_1\bar{\xi}_2 l_{19}\right]\varepsilon^2 + \ldots \tag{59}$$

The net maximum displacement ξ_m is

$$\xi_m = \varsigma_a + \eta_c = \varsigma(t_a, \tau_a) + \eta(T_c, \tau_c) \tag{60}$$

Substituting for terms in (60) from (57) and (59) we get

$$\xi_m = C_1\varepsilon + C_2\varepsilon^2 + \ldots \tag{61a}$$

where

$$C_1 = l_{22}; C_2 = \bar{\xi}_1 l_{23} + S^2 \bar{\xi}_2 l_{24} + k_1 \bar{\xi}_1^2 l_{12} + k_2 \bar{\xi}_2^2 \left[\frac{S}{R^2}\right]^2 l_{13} + R^2 \bar{\xi}_1 \bar{\xi}_2 S l_{19} \tag{61b}$$

$$l_{22} = 2\bar{\xi}_1 l_{10} + 2S\bar{\xi}_2 l_{20}; l_{23} = 2l_{11} + t_0 l_9; l_{24} = l_{21} + T_0 l_{18} \tag{61c}$$

As noted by $[1-3]$ and $[21]$,the condition for dynamic buckling is

$$\frac{d\lambda}{d\xi_m} = 0 \tag{62}$$

As in [21,22], applying the method of reversal of series of (61a), we get

$$\varepsilon = d_1\xi_m + d_2\xi_m^2 + \ldots \tag{63}$$

Substituting for ξ_m from (61a) in (63) and equating powers of orders of ε , we get

$$d_1 = \frac{1}{C_1}, d_2 = -\frac{C_2}{C_1^3} \tag{64}$$

The maximization in (62) is better done from (63), thus implementing (62) using (63) we have

$$\xi_m(\lambda_D) = \frac{C_1^2}{2C_2} \tag{65}$$

where, $\xi_m(\lambda_D)$ is the value of the net displacement at buckling. In determining the dynamic buckling load, we evaluate (63) at

$$\lambda = \lambda_D$$

to yield

$$\varepsilon = \xi_m(\lambda_D)[d_1 + d_2\xi_m]_{(\lambda=\lambda_D)} \tag{66}$$

On substituting for terms d_1 and d_2 from (64) and $\xi_m(\lambda_D)$ from (65) in (66) and simplify to get

$$\varepsilon\lambda_D = \frac{C_1}{4C_2} \tag{67}$$

The expansion of (67) gives [using (61b, c)]

$$\lambda_D = \frac{1}{4}\left[\frac{\omega_0}{\omega_1}\right]^2 \left[2\bar{\xi}_1 l_{10} + 2S\bar{\xi}_2 l_{20}\right] \left| \begin{bmatrix} \bar{\xi}_1 l_{23} + S^2 \bar{\xi}_2 l_{24} + k_1 \bar{\xi}_1^2 l_{12} + k_2 \bar{\xi}_2^2 \left[\frac{S}{R^2}\right]^2 l_{13} \\ + R^2 \bar{\xi}_1 \bar{\xi}_2 S l_{19} \end{bmatrix}^{-1} \right| \quad (68)$$

Here, (68) gives the formula for evaluating the dynamic buckling load λ_D , and is valid for $R \neq (1,2,Q,2Q,1-Q,1+Q)$ and $Q \neq (R,2R,1-R,1+R,0,2R-1)$

4. ANALYSIS OF RESULT

We note that the results display all the imperfection parameters stated in problems (3)-(5).This is unlike Danielson's problem in which the axisymmetric imperfection was neglected for easy solution. In fact, the method is such that we can adequately account for all modal imperfections allowed in the formulation .The contributions of the quadratic terms $k_1\xi_1^2, k_2\xi_2^2$ and the coupling term $\xi_1\xi_2$ are respectively given in the denominator of (68) by $k_1 \bar{\xi}_1^2 l_{11}, k_2 \bar{\xi}_2^2 \left[\frac{S}{R^2}\right] l_{13}$ and $R^2 \bar{\xi}_1 \bar{\xi}_2 S l_{19}$.

Thus if we assume that the axysymmetric imperfections are zero then $\bar{\xi}_1 = 0$, and the dynamic buckling load λ_D responsible for the buckling in this case is obtained from (68) as

$$\lambda_D = \frac{1}{4}\left[\frac{\omega_0}{\omega_1}\right]^2 \left[2S\bar{\xi}_2 l_{20}\right] \left| \left[S^2 \bar{\xi}_2 l_{24} + k_2 \bar{\xi}_2^2 \left[\frac{S}{R^2}\right]^2 l_{13}\right]^{-1} \right| \quad (69)$$

We note from (69), that, the effects of the coupling terms $\xi_1\xi_2$, $\xi_1\xi_0$ and the quadratic term $k_1\xi_1^2$ are zeros. The effect of the quadratic term $k_2\xi_2^2$ is non-zero and it is this term that dominates the buckling process. Neglecting $\bar{\xi}_1$ is sufficient to completely nullify the effect of ξ_1^2 where the converse is not necessarily the case.

However, if the non-axymmetric imperfections are neglected then $\bar{\xi}_2 = 0$, and the dynamic buckling load λ_D following (68) become

$$\lambda_D = \frac{1}{4}\left[\frac{\omega_0}{\omega_1}\right]^2 \left[2\bar{\xi}_1 l_{10}\right] \left| \left[\bar{\xi}_1 l_{23} + k_1 \bar{\xi}_1^2 l_{12}\right]^{-1} \right| \quad (70)$$

We deduce from (70), that, the effects of the coupling terms $\xi_1\xi_2$, $\xi_2\xi_0$ and the quadratic term $k_2\xi_2^2$ are again zeros. The effect of the quadratic term $k_1\xi_1^2$ is non-zero and this singular term is the only non-linear term that influences the buckling process. Neglecting $\bar{\xi}_2$ is sufficient to completely nullify the effect of ξ_2^2 where the converse is not necessarily the case.

The results also confirm that the only condition under which the effects of the coupling term $\bar{\xi}_1 \bar{\xi}_2$ would be felt is if none of the imperfection parameters in the shape of the mode coupling is neglected. In other word, is that nether the imperfection parameter $\bar{\xi}_1$ nor $\bar{\xi}_2$ should vanish for post dynamic buckling behavior of the structures. Once an imperfection is neglected the coupling effect of the mode that is in the shape of the neglected imperfection, with any other mode is neglected.

The graphical view of this phenomenon, we assume the following values. k_1 = 0.2, K_2 = 0.3,

$\bar{\xi}_1$ = 0.02, $\bar{\xi}_2$ = 0.03, α_1 = 0.01 and α_2 = 0.03. By varying $\bar{\xi}_2$ and α_2 while keeping $\bar{\xi}_1$ constant at 0.02 and α_1 = 0, the corresponding values of λ_D were computed from (68). The plots of dynamic buckling load against the imperfection parameter and light viscous damping of the discretized spherical cap are shown in Figs. 1 and 2 below.

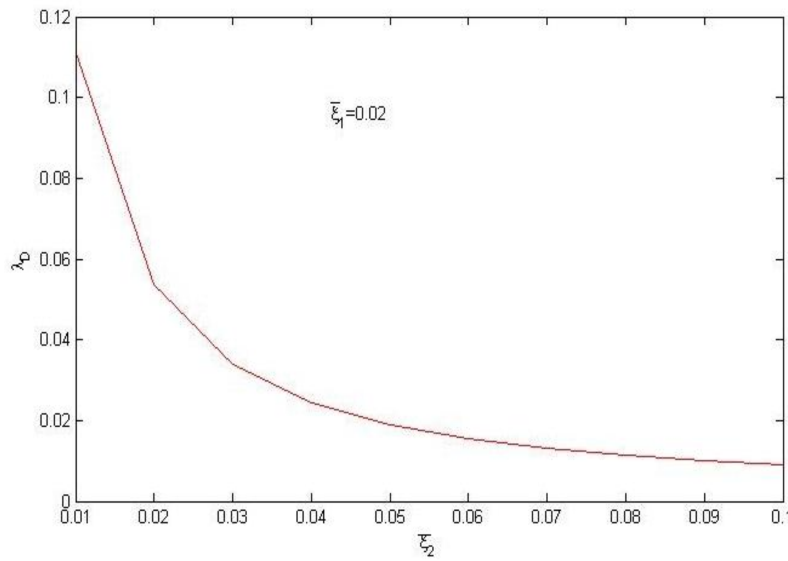

Fig. 1. Dynamic buckling load of a spherical cap against the imperfection parameter $\bar{\xi}_2$ ($\bar{\xi}_1 = 0.02$)

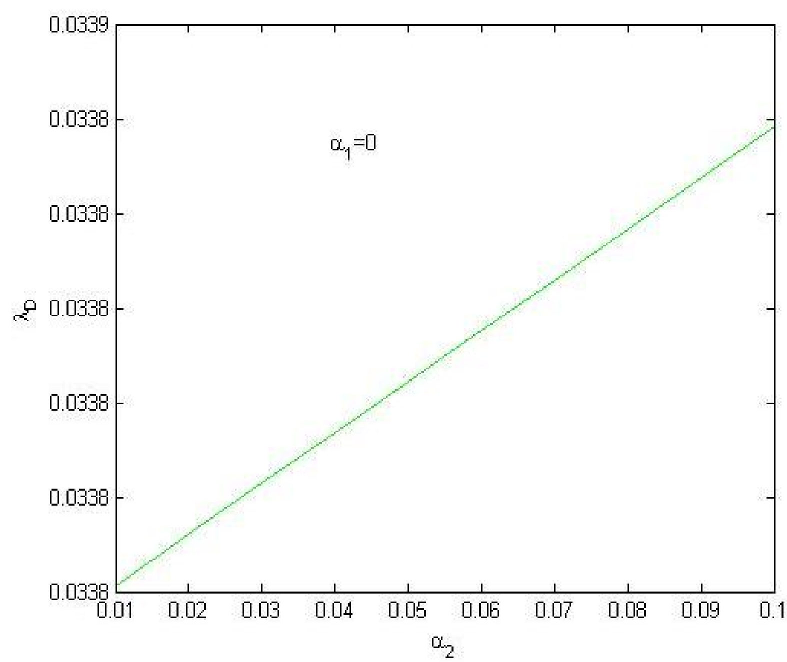

Fig. 2. Dynamic buckling load of a spherical cap against the light viscous damping α_2(α_1 = 0)

From Fig. 2 above we observe that dynamic buckling load increases with increased damping. Also in Fig. 1 dynamic buckling load increases if the structure is less imperfect, in other word, dynamic buckling load decreases with increased imperfection.

5. CONCLUSION

From the above discussions, we note that while neglecting the imperfection parameters $\bar{\xi}_1$ and $\bar{\xi}_2$ automatically implies, among other things, neglecting the effects of the non-linear terms $k_1\xi_1^2$ and $k_2\xi_2^2$ respectively. Also, we observe that the only condition under which the effect of the coupling term $\xi_1\xi_2$ would be felt, is when the imperfection parameters $\bar{\xi}_1$ and $\bar{\xi}_2$ are not equal to zero. Moreover, our results confirm those obtained by [23,24]. Finally, we notice that we can determine the value of the dynamic buckling load λ_D for whatever number of modal imperfections

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Hutchinson JW, Budiansky B. Dynamic buckling estimates. AIAAJ. 1966;4:525-530.
2. Budiansky B, Hutchinson JW. Dynamic buckling of imperfection- sensitive structure, proceedings of the XIth Inter. Congr. Applied Mech., Springer-Verlag, Berling; 1966.
3. Budiansky B. Dynamic buckling of elastic structures; Criteria and estimates, in, Dynamic stability of structures. Pergamons, New York; 1966.
4. Ohsaki M. Imperfection sensitivity of optimal, symmetric braced frames against buckling. Int. J. Non-Linear Mechanics. 2003;38(7):1103-1117.
5. Dumir PC, Dube GP, Mullick A. Axisymmetric static and dynamic buckling of laminated thick truncated conical cap. Int. J. Non-Linear Mechanics. 2003; 38(6):903-910.
6. Kardomateas GA, Simites GJ, Shem L, Li R. Buckling of sandwich wide columns, Int. J. Non-Linear Mechanics. 2002;37(7): 1239-1247.
7. Anwen W, Wenying. Characteristic-value analysis for plastic dynamic buckling of columns under elastoplastic compression waves. Int. J. Non-Linear Mechanics. 2003;38(5):615-628.
8. Rafteyiannis IG, Kounadis A. Interaction under step loading. Int. J. Non-Linear Mechanics. 2000;35(3):531-542.
9. Wei ZG, Yu JL, Batra RC. Dynamic buckling of thin cylindrical shells under axial impact. Int. J. Impact Engng. 2005; 32:572-592.
10. Batra RC, Wei ZG. (2005), Dynamic buckling of thin thermoviscoplastic rectangular plate. J. of Thin-Walled Structures. 2005;43:273-290.
11. Zhang T, Liu TG, Zhao Y, Luo JZ. Nonlinear dynamic buckling of stiffened plates under in-plane impact load. J. of Zhejiang University of Science. 2004;5(5): 609-617.
12. Danielson D. Dynamic buckling loads of imperfection sensitive structures from perturbation procedures. AIAAJ. 1969;7: 1506-1510.
13. Aksogan O. Sofiyev AV. Dynamic buckling of cylindrical shells with variable thickness subjected to a time-dependent external pressure varying as a power function of time, J. of Sound and Vibration. 2002; 254(4):693-703.
14. Schenk CA, Schueller GI. Buckling of cylindrical shells with random imperfections. Int. J. Non-Linear mechanics. 2003;38:1119-1132.
15. Wang A, Tian W. Twin characteristics-parameter solution under elastic compression waves. Int J. Solids Structures. 2002;39:861-877.
16. Wang A, Tian W. Twin characteristic parameter solutions of axisymmetric dynamic plastic buckling for cylindrical shells under axial compression waves. Int. J. Solids Structures. 2003;40:3157-3175.
17. Ette AM. On a two-small-parameter dynamic buckling of a lightly damped spherical cap trapped by a step load. Journal of the Nigerian Mathematics Society. 2007;23:7-26.
18. Ette AM. On a two-small-parameter dynamic stability of a lightly damped spherical shell pressurized by a harmonic excitation. Journal of the Nigerian

Association of Mathematical Physics. 2007;11:333-362.
19. Ette AM. On the buckling of lightly damped cylindrical shells modulated by a periodic load. Journal of the Nigerian Association of Mathematical Physics. 2006;10:327-344.
20. Ette AM. Perturbation technique on the dynamic stability of a damped cylindrical shell axially stressed by an impulse, Journal of the Nigerian Association of Mathematical Physics. 2008;12:103-120.
21. Amazigo JC. Buckling of stochastically imperfect columns of nonlinear elastic foundations. Quart. App. Math. 1973;31: 403.
22. Amazigo JC. Dynamic buckling of structures with random imperfections. Stochastic problem in Mechanics. Ed. H. Leipolz, University of Waterloo Press. 1974;243-254.
23. Ette AM. Dynamic buckling of imperfect spherical shell under an axial Impulse. Int. J. Non-Linear Mechanics. 1997;32(1):201-209.
24. Ette AM. Perturbation approached on the dynamic buckling of a lightly damped cylindrical shells modulated by a periodic load. Journal of the Nigerian Mathematics Society. 2009;28:97-135.

APPENDIX

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k1=0.2;
k2=0.3;
zi2bar=0.03;
zi1bar=0.02;
alpha1=0.01;
alpha2=0.03;
R=0.1;
Q=0.3;
S=(R/Q)^2;
L0=-1/Q^4;
L1=-1/(2*Q^2) + 1/(6*Q^2);
L2=0.5 + 1/(2*(Q^2-4*R^2))-2/(R^2*(Q^2-4*R^2));
L3=-1/R^4;
L4=1/(R^2*Q)^2 + 0.5*(-2/((R*Q)^2*(R^2-Q^2))-1/(Q*(R*Q)^2*(2*R+Q)) + 1/(Q*(R*Q)^2*(2*R-Q)));
L5=1/Q^2;
t0=pi/Q;
Bt0=pi/R;
phi=1/(2*Q)*(1+2*k1*zi1bar);
L6=sin(2*Q*t0)/(3*Q^2);
omega=1/(2*R)*(S-R^2*zi1bar/Q^2);
L7=Q^2*sin(2*R*t0)/(R^3*(Q^2-4*R^2));
L8=phi/Q;
L9=-alpha1/Q^2;
L10=1/Q^2;
L11=1/Q^4;
L12=1/Q^2-1/(3*Q^4);
L13=-1 -1/(2*(Q^2-4*R^2)) + 1/(R^2*(Q^2-4*R^2))-Q^2*(1/(Q^2-4*R^2))+ 2/(Q^2-R^2);
L14=-cos(R*Bt0)/R^2;
L15=-R*L3*sin(R*Bt0);
L16=-R^3*S*L4*sin(R*Bt0)-(R^2/2)*(2*sin(Q*Bt0)/(Q*R^2*(R^2-Q^2))-
cos(Q*Bt0)/((R*Q)^2*(2*R+Q))-(Q+R)*sin(Q+R)*Bt0/(Q*(R*Q)^2*(2*R+Q))-(Q-R)*sin(Q-
R)*Bt0/(Q*(R*Q)^2*(2*R-Q)));
L17=omega/R;
L18=-alpha2/R^2;
L19=-L4 + 0.5*(2*cos(Q*Bt0)/(Q^2*R^2*(R^2-Q^2)) + cos(Q+R)*Bt0/(Q*(R*Q)^2*(2*R+Q)) +
cos(Q-R)*Bt0/(Q*(R*Q)^2*(2*R-Q)));
L20=1/R^2;
L21=1/R^4;
L22=2*zi1bar*L10 + 2*S*zi2bar*L20;
L23=2*L11 + t0*L9;
L24=L21+Bt0*L18;
c1=L22;
c2=zi1bar*L23+S^2*zi2bar*L24+k1*zi1bar^2*L12 +
k2*zi2bar^2*(S/R^2)^2*L13+R^2*zi1bar*zi2bar*S*L19;
LambdaD=c1/(4*c2*Q^2);
```

z=LambdaD;

INTERPRETATION OF VARIBLES

zi1bar = $\bar{\xi}_1$, zi2bar = $\bar{\xi}_2$, alpha1 = α_1 , alpha2 = α_2 , t0 = t_0 , Bt0 = T_0 , phi = φ , Omega = Φ , pi= π , LambdaD = λ_D .

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