Open Journal of Mathematical Sciences

PSR Press

Article

Generalizations of generating functions for basic hypergeometric orthogonal polynomials

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Academic Editor: Ljubisa Kocinac
Received: 7 September 2021; Accepted: 1 September 2022; Published: 19 November 2022.

Abstract: We derive generalized generating functions for basic hypergeometric orthogonal polynomials by applying connection relations with one extra free parameter to them. In particular, we generalize generating functions for the continuous q-ultraspherical/Rogers, little q-Laguerre/Wall, and q-Laguerre polynomials. Depending on what type of orthogonality these polynomials satisfy, we derive corresponding definite integrals, infinite series, bilateral infinite series, and q-integrals.

Keywords: Basic hypergeometric series; Basic hypergeometric orthogonal polynomials; Generating functions; Connection coefficients; Eigenfunction expansions; Definite integrals; Infinite series; Bilateral infinite series; q-integrals.

MSC: 33D45; 05A15; 33D15; 33C45; 33C20; 34L10; 30E20.

1. Introduction

In the context of generalized hypergeometric orthogonal polynomials Cohl applied in [1, (2.1)] a series rearrangement technique which produces a generalization of the generating function for the Gegenbauer polynomials. We have since demonstrated that this technique is valid for a larger class of hypergeometric orthogonal polynomials. For instance, in [2] we applied this same technique to the Jacobi polynomials, and in [3], we extended this technique to many generating functions for the Jacobi, Gegenbauer, Laguerre, and Wilson polynomials.

The series rearrangement technique combines a connection relation with a generating function, resulting in a series with multiple sums. The order of summations are then rearranged and the result often simplifies to produce a generalized generating function whose coefficients are given in terms of generalized or basic hypergeometric functions. This technique is especially productive when using connection relations with one extra free parameter, since the relation is most often a product of shifted factorials (Pochhammer symbols) and q-shifted factorials (q-Pochhammer symbols).

Basic hypergeometric orthogonal polynomials with more than one extra free parameter, such the Askey–Wilson polynomials, have multi-parameter connection relations. These connection relations are in general given by single or multiple summation expressions. For the Askey–Wilson polynomials, the connection relation with four extra free parameters is given as a basic double hypergeometric series. The fact that the four extra free parameter connection coefficient for the Askey–Wilson polynomials is given by a double sum was known to Askey and Wilson as far back as 1985 (see [4, p. 444]). When our series rearrangement technique is applied to cases with more than one extra free parameter, the resulting coefficients of the generalized generating function are rarely given in terms of a basic hypergeometric series. The more general problem of generalized generating functions with more than one extra free parameter requires the theory of multiple basic hypergeometric series and is not treated in this paper.

In this paper, we apply this technique to generalize generating functions for basic hypergeometric orthogonal polynomials in the q-analog of the Askey scheme [5, Chapter 14]. In §2, we give some preliminary material which is used in the remainder of the paper. In §3, we present generalizations of the continuous q-ultraspherical/Rogers polynomials. In §4, we present generalizations of the little q-Laguerre polynomials. In §5, we present generalizations of the q-Laguerre polynomials. In §6, we have also computed new definite integrals, infinite series, and Jackson integrals (hereafter q-integrals) corresponding to our generalized generating function expansions using orthogonality for the studied basic hypergeometric orthogonal polynomials.

Note that one important class of hypergeometric orthogonal polynomial generating functions which does not seem amenable to our series rearrangement technique are bilinear generating functions. The existence of an extra orthogonal polynomial in the generating function, produces multiple summation expressions via the introduction of connection relations for one or both of the polynomials with the sums being formidable to evaluate in closed form.

2. Preliminaries

Define $\mathbb{N}_0 := \{0\} \cup \mathbb{N} := \{0\} \cup \{1, 2, 3, \ldots\}$. Throughout the paper, we will adopt the following notation to indicate sequential positive and negative elements, in a list of elements, namely

$$\pm a := \{a, -a\}.$$

If $\pm$ appears in an expression, but not in a list, it is to be treated as normal. In order to obtain our derived identities, we rely on properties of the q-shifted factorial. The shifted factorial and q-shifted factorial are defined for all $n \in \mathbb{N}_0$, $a \in \mathbb{C}$ such that

$$\begin{aligned} &(a)_0 := 1, \quad (a)_n := (a)(a+1)\cdots(a+n-1), \quad n \in \mathbb{N}, \\ &(a;q)_0 := 1, \quad (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \in \mathbb{N}. \end{aligned} \tag{1}$$

Note $(a)_b := \Gamma(a+b)/\Gamma(a)$ for all $a, b \in \mathbb{C}$, $a+b \notin -\mathbb{N}_0$. Also define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \tag{2}$$

where $0 < |q| < 1$, $a \in \mathbb{C}$. We will also use the common notational product conventions

$$\begin{aligned} &(a_1, \ldots, a_k)_b := (a_1)_b \cdots (a_k)_b, \\ &(a_1, \ldots, a_k; q)_b := (a_1;q)_b \cdots (a_k;q)_b, \end{aligned}$$

where $a_1, a_2, \ldots, a_k, b \in \mathbb{C}$. We define the q-factorial as [6, (1.2.44)]

$$[0]_q! := 1, \; [n]_q! := [1]_q[2]_q \cdots [n]_q, \quad n \in \mathbb{N},$$

where the q-number is defined as [5, (1.8.1)]

$$[z]_q := \frac{1-q^z}{1-q}, \quad z \in \mathbb{C},$$

with $q \in \mathbb{C}$, $q \neq 1$. Note that $[n]_q! = (q;q)_n/(1-q)^n$.

The following properties for the q-shifted factorial can be found in [5, (1.8.7), (1.8.10-11), (1.8.14), (1.8.19), (1.8.21-22)], namely for appropriate values of a and $n, k \in \mathbb{N}_0$,

$$(a^{-1};q)_n = \frac{(-1)^n}{a^n} q^{\binom{n}{2}} (a;q^{-1})_n, \tag{3}$$

$$(a;q)_{n+k} = (a;q)_k (aq^k;q)_n = (a;q)_n (aq^n;q)_k, \tag{4}$$

$$(aq^n;q)_k = \frac{(a;q)_k}{(a;q)_n} (aq^k;q)_n, \tag{5}$$

$$(aq^{-n};q)_k = q^{-nk} \frac{(q/a;q)_n}{(q^{1-k}/a;q)_n} (a;q)_k, \tag{6}$$

$$(a;q)_{2n} = (a, aq; q^2)_n = (\pm\sqrt{a}, \pm\sqrt{qa}; q)_n, \tag{7}$$

$$(a^2;q^2)_n = (\pm a;q)_n. \tag{8}$$

Observe that by using (1), (7) and (8), we get

$$(aq^n;q)_n = \frac{(\pm\sqrt{a}, \pm\sqrt{aq};q)_n}{(a;q)_n} = \frac{(a;q)_{2n}}{(a;q)_n}. \tag{9}$$

Lemma 1. *Let $q, \alpha, \beta \in \mathbb{C}$, $0 < |q| < 1$. Then*

$$\lim_{q \uparrow 1^-} \frac{(q^\alpha;q)_\beta}{(1-q)^\beta} = (\alpha)_\beta. \tag{10}$$

Proof. See [7, Lemma 2.2]. □

We also take advantage of the q-binomial [5, (1.11.1)] and binomial [5, (1.5.1)] theorems, $a \in \mathbb{C}$, $|z| < 1$, respectively $|q| < 1$,

$${}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty},$$

where we have used (2), and

$${}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; q, z\right) = (1-z)^{-a}.$$

The basic hypergeometric series, which we will often use, is defined for $|z| < 1$, $0 < |q| < 1$, $s, r \in \mathbb{N}_0$, $a_l, b_j \in \mathbb{C}$, $b_j \notin -\mathbb{N}_0$, $l, j \in \mathbb{N}_0$, $0 \le l \le r$, $0 \le j \le s$, as [5, (1.10.1)]

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r;q)_k}{(q, b_1, \ldots, b_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k. \tag{11}$$

Note that [5, p. 15]

$$\lim_{q \uparrow 1^-} {}_r\phi_s\left(\begin{matrix} q^{a_1}, \ldots, q^{a_r} \\ q^{b_1}, \ldots, q^{b_s} \end{matrix}; q, (q-1)^{1+s-r} z\right) = {}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} \frac{z^k}{k!}, \tag{12}$$

where ${}_rF_s$ is the generalized hypergeometric series [8, Chapter 16].

Let us prove some inequalities that we will later use.

Lemma 2. *Let $j \in \mathbb{N}$, $k, n \in \mathbb{N}_0$, $z \in \mathbb{C}$, $\Re u > 0$, $v \ge 0$, and $0 < |q| < 1$. Then*

$$\left|(q^u;q)_j\right| \ge \left|(1-|q|)[\Re u]_q\right| \left|(q;q)_{j-1}\right|, \tag{13}$$

$$\left|\frac{(q^u;q)_n}{(q;q)_n}\right| \le \left|[n+1]_q^{u+1}\right|, \tag{14}$$

$$\left|\frac{(q^{v+k};q)_n}{(q^{u+k};q)_n}\right| \le \left|\frac{[n+1]_q^{v+1}}{[\Re(u)]_q}\right|. \tag{15}$$

Proof. See [7, Lemma 2.3]. □

For a family of orthogonal polynomials $(P_n(x;\mathbf{a}))$, where $\mathbf{a}$, $\mathbf{b}$, are sets of free parameters, define $a_n(\mathbf{a})$, $c_{k,n}(\mathbf{a};\mathbf{b})$ as follows. A generating function for these orthogonal polynomials is defined as

$$f(x,t,\mathbf{a}) = \sum_{n=0}^{\infty} a_n(\mathbf{a}) P_n(x;\mathbf{a}) t^n,$$

and a connection relation for these orthogonal polynomials is defined as

$$P_n(x;\mathbf{a}) = \sum_{k=0}^{n} c_{k,n}(\mathbf{a};\mathbf{b}) P_k(x;\mathbf{b}).$$

3. The continuous q-ultraspherical/Rogers polynomials

The continuous q-ultraspherical/Rogers polynomials are defined as [5, (14.10.17)]

$$C_n(x;\beta|q) := \frac{(\beta;q)_n}{(q;q)_n} e^{in\theta} {}_2\phi_1\left(\begin{matrix} q^{-n}, \beta \\ \beta^{-1}q^{1-n} \end{matrix}; q, q\beta^{-1}e^{-2i\theta}\right), \qquad x = \cos\theta.$$

By starting with generating functions for the continuous q-ultraspherical/Rogers polynomials [5, (14.10.27–33)], we derive generalizations using the connection relation for these polynomials, namely [4, (13.3.1)]

$$C_n(x;\beta\,|\,q) = \frac{1}{1-\gamma}\sum_{k=0}^{\lfloor n/2\rfloor} \frac{(1-\gamma q^{n-2k})\gamma^k(\beta\gamma^{-1};q)_k(\beta;q)_{n-k}}{(q;q)_k(q\gamma;q)_{n-k}} C_{n-2k}(x;\gamma|q). \tag{16}$$

Theorem 1. *Let $x \in [-1,1]$, $0 < |\beta|, |\gamma|, |q| < 1$, $|t\beta|(1-|q|)^2 < 1-|\beta|^2$. Then*

$$(te^{-i\theta};q)_\infty\, {}_2\phi_1\left(\begin{matrix} \beta, \beta e^{2i\theta} \\ \beta^2 \end{matrix}; q, te^{-i\theta}\right) = \sum_{n=0}^{\infty} \frac{(\beta;q)_n\, q^{\binom{n}{2}}(-\beta t)^n}{(\gamma, \beta^2;q)_n} C_n(x;\gamma|q)$$
$$\times\, {}_2\phi_5\left(\begin{matrix} \beta\gamma^{-1}, \beta q^n \\ \gamma q^{n+1}, \pm\beta q^{n/2}, \pm\beta q^{(n+1)/2} \end{matrix}; q, \gamma(\beta t)^2 q^{2n+1}\right). \tag{17}$$

Proof. A generating function for continuous q-ultraspherical/Rogers polynomials can be found in [5, (14.10.29)]

$$(te^{-i\theta};q)_\infty\, {}_2\phi_1\left(\begin{matrix} \beta, \beta e^{2i\theta} \\ \beta^2 \end{matrix}; q, te^{-i\theta}\right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-\beta t)^n}{(\beta^2;q)_n} C_n(x;\beta|q). \tag{18}$$

Start with (18), inserting (16), shifting the n index by $2k$, reversing the order of summation and using (4) through (11), and by noting

$$\binom{n+2k}{2} = \binom{n}{2} + 4\binom{k}{2} + (2n+1)k.$$

Define

$$\langle \mathbf{u}, f(x)\rangle := \int_{-1}^{1} f(x)\frac{w_R(x;\beta|q)}{\sqrt{1-x^2}}\,dx, \tag{19}$$

where $w_R : (-1,1) \to [0,\infty)$ is the weight function defined by

$$w_R(x;\beta|q) := \left|\frac{(e^{2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty}\right|^2. \tag{20}$$

Moreover, since ([5, (14.10.18]) $|C_n(x;\beta|q)| \le K_1[n+1]^{\sigma_1}$, then

$$|\langle \mathbf{u}, C_n(x;\beta|q)C_k(x;\beta|q)\rangle| \le K_1^2[n+1]^{2\sigma_1},$$

and

$$|\langle \mathbf{u}, C_n(x;\beta|q)C_n(x;\beta|q)\rangle| = \left|\frac{(\beta^2;q)_n(1-\beta)}{(q;q)_n(1-\beta q^n)}\right| \ge \left|\frac{[\Re(2b)]_q[\Re(b)]_q}{[2]_q^{b+2n+1}}\right| \ge \frac{K_2}{(1-|q|)^{2n}},$$

where $q^b = \beta$, so

$$|c_{k,n}| = \left|\frac{\langle \mathbf{u}, C_n(x;\beta|q)C_k(x;\beta|q)\rangle}{\langle \mathbf{u}, C_n(x;\beta|q)C_n(x;\beta|q)\rangle}\right| \le \frac{K_1^2}{K_2}[n+1]^{2\sigma_1}(1-|q|)^{2n},$$

$|a_n| \le (|t\beta|/(1-|\beta|^2))^n$. Therefore

$$\sum_{n=0}^{\infty}|a_n|\sum_{k=0}^{\lfloor n/2\rfloor}|c_{k,n}||C_k(x;\beta|q)| \le \frac{K_1^3}{K_2}\sum_{n=0}^{\infty}\frac{(1-|q|)^{2n}|t|^n|\beta|^2}{(1-|\beta|^2)^n}(n+1)^{3\sigma_1+1} < \infty,$$

and the result follows. □

Corollary 2. *Let* $x \in [-1,1]$, $|t| < 1$, $\beta, \gamma \in (-1,\infty) \setminus \{0,1\}$, $0 < |q| < 1$. *Then*

$$e^{xt}\,{}_0F_1\left(\begin{matrix} - \\ \beta+\frac{1}{2} \end{matrix}; \frac{(x^2-1)t^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(\beta)_n t^n}{(\gamma,2\beta)_n} C_n^{\gamma}(x)\,{}_2F_3\left(\begin{matrix} \beta-\gamma, \beta+n \\ \gamma+n+1, \beta+\frac{n}{2}, \beta+\frac{n+1}{2} \end{matrix}; \frac{t^2}{4}\right). \tag{21}$$

Proof. In (17), transform $\beta \mapsto q^\beta$, $\gamma \mapsto q^\gamma$, $t \mapsto (1-q)t$, and take the limit as $q \uparrow 1^-$. Using the definition of the q-exponential function [5, (1.14.2)] $\mathrm{E}_q(z) := (-z;q)_\infty$, $\lim_{q\uparrow 1^-} \mathrm{E}_q((1-q)z) = e^z$, and that the ${}_2\phi_1$ becomes a Kummer confluent hypergeometric functions ${}_1F_1$ with argument $-2it\sin\theta$. Representing this as a Bessel function of the first kind using [8, (10.16.5)], and then using [8, (10.2.2)], the left-hand side follows. The $q \uparrow 1^-$ limit on the right-hand side is straightforward. □

Theorem 3. *Let* $x \in [-1,1]$, $|t|(1-|q|)^2 < 1-|\beta|^2$, $0 < |\beta|, |\gamma|, |q| < 1$. *Then*

$$\frac{1}{(te^{i\theta};q)_\infty}\,{}_2\phi_1\left(\begin{matrix} \beta, \beta e^{2i\theta} \\ \beta^2 \end{matrix}; q, te^{-i\theta}\right) = \sum_{n=0}^{\infty} \frac{(\beta;q)_n t^n}{(\gamma,\beta^2;q)_n} C_n(x;\gamma|q)\,{}_6\phi_5\left(\begin{matrix} \beta\gamma^{-1}, \beta q^n, 0,0,0,0 \\ \gamma q^{n+1}, \pm\beta q^{\frac{n}{2}}, \pm\beta q^{\frac{n+1}{2}} \end{matrix}; q, \gamma t^2\right). \tag{22}$$

Proof. A generating function for the continuous q-ultraspherical/Rogers polynomials can be found in [5, (14.10.28)]

$$\frac{1}{(te^{i\theta};q)_\infty}\,{}_2\phi_1\left(\begin{matrix} \beta, \beta e^{2i\theta} \\ \beta^2 \end{matrix}; q, te^{-i\theta}\right) = \sum_{n=0}^{\infty} \frac{C_n(x;\beta|q)}{(\beta^2;q)_n} t^n. \tag{23}$$

The proof follows as above by starting with (23), inserting (16), shifting the n index by $2k$, reversing the order of summation and using (4) through (11). □

Remark 1. The $q \uparrow 1^-$ limit of (22) can also be shown to be the same as (21), by using the transformation $x \mapsto -x$. The proof of this is the same as the proof of Corollary 2, except instead use the definition of the q-exponential function [5, (1.14.1)] $e_q(z) := 1/(z;q)_\infty$, $\lim_{q\uparrow 1^-} e_q((1-q)z) = e^z$. Of course, the same is true for the $q \uparrow 1^-$ limits of the original generating functions [5, (14.10.28–29)], which both are analogues of [5, (9.8.31)], and are equivalent under the transformation $x \mapsto -x$.

Theorem 4. *Let* $x \in [-1,1]$, $|t|(1-|q|)^2 < 1$, $0 < |\beta|, |\gamma|, |q| < 1$. *Then*

$$\begin{aligned} &\frac{(\gamma te^{i\theta};q)_\infty}{(te^{i\theta};q)_\infty}\,{}_3\phi_2\left(\begin{matrix} \gamma, \beta, \beta e^{2i\theta} \\ \beta^2, \gamma te^{i\theta} \end{matrix}; q, te^{-i\theta}\right) \\ &\quad = \sum_{n=0}^{\infty} \frac{(\beta,\gamma;q)_n t^n}{(\alpha,\beta^2;q)_n} C_n(x;\alpha|q)\,{}_6\phi_5\left(\begin{matrix} \beta/\alpha, \beta q^n, \pm(\gamma q^n)^{\frac{1}{2}}, \pm(\gamma q^{n+1})^{\frac{1}{2}} \\ \alpha q^{n+1}, \pm\beta q^{n/2}, \pm\beta q^{(n+1)/2} \end{matrix}; q, \alpha t^2\right). \end{aligned} \tag{24}$$

Proof. A generating function for the continuous q-ultraspherical/Rogers polynomials can be found in [5, (14.10.33)]

$$\frac{(\gamma te^{i\theta};q)_\infty}{(te^{i\theta};q)_\infty}\,{}_3\phi_2\left(\begin{matrix} \gamma, \beta, \beta e^{2i\theta} \\ \beta^2, \gamma te^{i\theta} \end{matrix}; q, te^{-i\theta}\right) = \sum_{n=0}^{\infty} \frac{(\gamma;q)_n}{(\beta^2;q)_n} C_n(x;\beta|q) t^n, \tag{25}$$

where $\gamma \in \mathbb{C}$. Substitute (16) into the generating function (25), reverse the order of summation as above, shift the n index by $2k$, using (4) through (11), completes the proof. □

Theorem 5. *Let* $x \in [-1,1]$, $(1-|q|)^2(1+|\sqrt{q}||\beta|)|t| < (1-|q||\beta|)$, $0 < |\beta|, |\gamma|, |q| < 1$. *Then*

$$\begin{aligned} &{}_2\phi_1\left(\begin{matrix} \pm\beta^{\frac{1}{2}} e^{i\theta} \\ -\beta \end{matrix}; q, te^{-i\theta}\right) {}_2\phi_1\left(\begin{matrix} \pm(q\beta)^{\frac{1}{2}} e^{-i\theta} \\ -q\beta \end{matrix}; q, te^{i\theta}\right) = \sum_{n=0}^{\infty} \frac{(\beta, \pm\beta q^{\frac{1}{2}};q)_n t^n}{(\gamma,\beta^2,-q\beta;q)_n} C_n(x;\gamma|q) \\ &\quad \times {}_{10}\phi_9\left(\begin{matrix} \beta\gamma^{-1}, \beta q^n, \pm(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \pm(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}}, \pm i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \pm i(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}} \\ \gamma q^{n+1}, \pm\beta q^{n/2}, \pm\beta q^{(n+1)/2}, \pm i(\beta q^{n+1})^{\frac{1}{2}}, \pm i(\beta q^{n+2})^{\frac{1}{2}} \end{matrix}; q, \gamma t^2\right). \end{aligned} \tag{26}$$

Proof. A generating function for the continuous q-ultraspherical/Rogers polynomials can be found in [5, (14.10.31)]

$$ {}_2\phi_1\left(\begin{matrix}\pm\beta^{\frac{1}{2}}e^{i\theta} \\ -\beta\end{matrix};q,te^{-i\theta}\right){}_2\phi_1\left(\begin{matrix}\pm(q\beta)^{\frac{1}{2}}e^{-i\theta} \\ -q\beta\end{matrix};q,te^{i\theta}\right)=\sum_{n=0}^{\infty}\frac{(\pm\beta q^{\frac{1}{2}};q)_n}{(\beta^2,-q\beta;q)_n}C_n(x;\beta|q)\,t^n. \tag{27} $$

We substitute (16) into the generating function (27), switch the order of the summation, shift the n index by $2k$ and using (4) through (11), produces

$$ |a_n|\leq K_3\frac{(1+|\sqrt{q}||\beta|)^n|t|^n}{(1-|q||\beta|)^n}[n+1]^{\sigma_2}. $$

Therefore the theorem holds. □

Theorem 6. *Let* $x\in[-1,1]$, $(1-|q|)^2(1+|\beta||\sqrt{q}|)(1+|\beta|)|t|<(1-|\sqrt{q}||\beta|)(1-|\beta|^2)$, $0<|\beta|,|\gamma|,|q|<1$. *Then*

$$ {}_2\phi_1\left(\begin{matrix}\beta^{\frac{1}{2}}e^{i\theta},(q\beta)^{\frac{1}{2}}e^{i\theta} \\ \beta q^{\frac{1}{2}}\end{matrix};q,te^{-i\theta}\right){}_2\phi_1\left(\begin{matrix}-\beta^{\frac{1}{2}}e^{-i\theta},-(q\beta)^{\frac{1}{2}}e^{-i\theta} \\ \beta q^{\frac{1}{2}}\end{matrix};q,te^{i\theta}\right)=\sum_{n=0}^{\infty}\frac{(\pm\beta,-\beta q^{\frac{1}{2}};q)_n t^n}{(\gamma,\beta^2,\beta q^{\frac{1}{2}};q)_n}C_n(x;\gamma|q) $$
$$ \times{}_{10}\phi_9\left(\begin{matrix}\beta\gamma^{-1},\beta q^n,\pm i(\beta q^n)^{\frac{1}{2}},\pm i(\beta q^{n+1})^{\frac{1}{2}},\pm i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}},\pm i(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}} \\ \gamma q^{n+1},\pm\beta q^{n/2},\pm\beta q^{(n+1)/2},\pm(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}},\pm(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}}\end{matrix};q,\gamma t^2\right). \tag{28} $$

Proof. We start with the generating function for the continuous q-ultraspherical/Rogers polynomials [5, (14.10.30)]

$$ {}_2\phi_1\left(\begin{matrix}\beta^{\frac{1}{2}}e^{i\theta},(q\beta)^{\frac{1}{2}}e^{i\theta} \\ \beta q^{\frac{1}{2}}\end{matrix};q,te^{-i\theta}\right){}_2\phi_1\left(\begin{matrix}-\beta^{\frac{1}{2}}e^{-i\theta},-(q\beta)^{\frac{1}{2}}e^{-i\theta} \\ \beta q^{\frac{1}{2}}\end{matrix};q,te^{i\theta}\right)=\sum_{n=0}^{\infty}\frac{(-\beta,-\beta q^{\frac{1}{2}};q)_n}{(\beta^2,\beta q^{\frac{1}{2}};q)_n}C_n(x;\beta|q)t^n. \tag{29} $$

Using the connection relation (16) in (29), reversing the orders of the summation, shifting the n index by $2k$, and using (4) through (11), obtains the result

$$ |a_n|\leq\frac{(1+|\beta||\sqrt{q}|)^n(1+|\beta|)^n|t|^n}{(1-|\beta|^2)^n(1-|\sqrt{q}||\beta|)^n}. $$

Therefore the theorem holds. □

Theorem 7. *Let* $x\in[-1,1]$, $(1-|q|)^2(1+|\beta|)|t|<(1-|\sqrt{q}||\beta|)$, $0<|\beta|,|\gamma|,|q|<1$. *Then*

$$ {}_2\phi_1\left(\begin{matrix}\beta^{\frac{1}{2}}e^{i\theta},-(q\beta)^{\frac{1}{2}}e^{i\theta} \\ -\beta q^{\frac{1}{2}}\end{matrix};q,te^{-i\theta}\right){}_2\phi_1\left(\begin{matrix}(q\beta)^{\frac{1}{2}}e^{-i\theta},-\beta^{\frac{1}{2}}e^{-i\theta} \\ -\beta q^{\frac{1}{2}}\end{matrix};q,te^{i\theta}\right)=\sum_{n=0}^{\infty}\frac{(\pm\beta,\beta q^{\frac{1}{2}};q)_n t^n}{(\gamma,\beta^2,-\beta q^{\frac{1}{2}};q)_n}C_n(x;\gamma|q) $$
$$ \times{}_{10}\phi_9\left(\begin{matrix}\beta\gamma^{-1},\beta q^n,\pm i(\beta q^n)^{\frac{1}{2}},\pm i(\beta q^{n+1})^{\frac{1}{2}},\pm(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}},\pm(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}} \\ \gamma q^{n+1},\pm\beta q^{n/2},\pm\beta q^{(n+1)/2},\pm i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}},\pm i(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}}\end{matrix};q,\gamma t^2\right). \tag{30} $$

Proof. A generating function for the continuous q-ultraspherical/Rogers polynomials can be found in [5, (14.10.32)]

$$ {}_2\phi_1\left(\begin{matrix}-\beta,(\beta q)^{\frac{1}{2}} \\ -\beta q^{\frac{1}{2}}\end{matrix};q,te^{-i\theta}\right){}_2\phi_1\left(\begin{matrix}(\beta q)^{\frac{1}{2}}e^{-i\theta},-\beta^{\frac{1}{2}}e^{-i\theta} \\ -\beta q^{\frac{1}{2}}\end{matrix};q,te^{i\theta}\right)=\sum_{n=0}^{\infty}\frac{(-\beta,\beta q^{\frac{1}{2}};q)_n}{(\beta^2,-\beta q^{\frac{1}{2}};q)_n}C_n(x;\beta|q)\,t^n. \tag{31} $$

Similar to the proof of (28), we substitute (16) into the generating function (31), switch the order of the summation, shift the n sum by $2k$, and use (4) through (11), obtaining the result

$$ |a_n|\leq K_4\frac{(1+|\beta|)^n|t|^n}{(1-|\sqrt{q}||\beta|)^n}[n+1]^{\sigma_3}. $$

Therefore the theorem holds. □

4. The little q-Laguerre/Wall polynomials

The little q-Laguerre/Wall polynomials are defined as [5, (14.20.1)]

$$p_n(x;a|q) := {}_2\phi_1\left(\begin{matrix} q^{-n},0 \\ aq \end{matrix};q,qx\right) = \frac{1}{(a^{-1}q^{-n};q)_n}\,{}_2\phi_0\left(\begin{matrix} q^{-n},x^{-1} \\ - \end{matrix};q,\frac{x}{a}\right).$$

The connection relation for little q-Laguerre/Wall polynomials can be obtained by Exercise 1.33 in [6] and using the specialization formula which connects the little q-Laguerre/Wall polynomials with the little q-Jacobi polynomials, namely [5, p. 521] $p_n(x;a|q) = p_n(x;a,0|q)$.

Theorem 8. *Let $0 < |aq|, |bq|, |q| < 1$. Then the connection relation for the little q-Laguerre/Wall polynomials is given by*

$$p_n(x;a|q) = \frac{q^{-\binom{n}{2}}}{(qa;q)_n}\sum_{j=0}^{n}\frac{q^{\binom{j}{2}+n(n-j)}(-a)^{n-j}(q^{n-j+1},qb;q)_j(bq^{1+j-n}/a;q)_{n-j}}{(q;q)_j}\,p_j(x;b|q). \tag{32}$$

By starting with the generating function for the little q-Laguerre/Wall polynomials [5, (14.20.11)], we derive generalizations using the connection relation for these polynomials.

Theorem 9. *Let $0 < |aq|, |bq|, |q| < 1$, $|t| < \min\{(1-q)(1-aq)/a, 1\}$. Then*

$$\frac{(t;q)_\infty}{(xt;q)_\infty}\,{}_0\phi_1\left(\begin{matrix} - \\ aq \end{matrix};q,aqxt\right) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(-t)^n(bq;q)_n}{(q;q)_n(aq;q)_n}\,p_n(x;b|q)\,{}_1\phi_1\left(\begin{matrix} a/b \\ aq^{n+1} \end{matrix};q,bq^{n+1}t\right). \tag{33}$$

Proof. We start with the generating function for little q-Laguerre/Wall polynomials found in [5, (14.20.11)]

$$\frac{(t;q)_\infty}{(xt;q)_\infty}\,{}_0\phi_1\left(\begin{matrix} - \\ aq \end{matrix};q,aqxt\right) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}p_n(x;a|q)t^n. \tag{34}$$

Using the connection relation (32) in (34), reversing the orders of the summations, shifting the n index by j, and using (4) through (11), obtains the desired result since $|a_n| = |t|^n/(1-q)^n$, $|c_{n,k}| \le K_8[n+1]^{\sigma_6}$, and

$$|p_n(x;a|q)| \le |a|^n[n+1]^{\sigma_7}/(1-|aq|)^n \le a^n(n+1)^{\sigma_7}/(1-aq)^n, \tag{35}$$

where σ_6 and σ_7 are independent of n implies

$$\sum_{n=0}^{\infty}|a_n|\sum_{k=0}^{n}|c_{k,n}||p_k(x;a|q)| < \infty.$$

Therefore the theorem holds. □

5. The q-Laguerre polynomials

The q-Laguerre polynomials are defined as [5, (14.21.1)]

$$L_n^{(\alpha)}(x;q) := \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix};q,-q^{n+\alpha+1}x\right) = \frac{1}{(q;q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n},-x \\ 0 \end{matrix};q,q^{n+\alpha+1}\right).$$

Theorem 10. *Let $\alpha,\beta \in (-1,\infty)$, $0 < |q| < 1$. The connection relation for the q-Laguerre polynomials is given as*

$$L_n^{(\alpha)}(x;q) = \frac{q^{n(\alpha-\beta)}}{(q;q)_n}\sum_{j=0}^{n}(-1)^{n-j}q^{\binom{n-j}{2}}(q^{n-j+1};q)_j(q^{j-n+\beta-\alpha+1};q)_{n-j}\,L_j^{(\beta)}(x;q). \tag{36}$$

Proof. One could obtain the above result by following an analogous proof as applied to the little q-Laguerre/Wall polynomials. Nevertheless the result follows by using the relation between the little q-Laguerre/Wall and the q-Laguerre polynomials [5, p. 521]. □

By starting with generating functions for the q-Laguerre polynomials [5, (14.21.14–16)], we derive generalizations of these generating functions using the connection relation for q-Laguerre polynomials (36). Note however that the generating function for the q-Laguerre polynomials [5, (14.21.13)] remains unchanged when one applies the connection relation (36).

Theorem 11. *Let $\alpha, \beta \in (-1, \infty)$, $0 < |q| < 1$, $|t| < (1 - q^{\alpha+1})(1 - q)$. Then*

$$\frac{1}{(t;q)_\infty} {}_0\phi_1 \left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix} ; q, -xtq^{\alpha+1} \right) = \sum_{n=0}^{\infty} \frac{(q^{\alpha-\beta}t)^n L_n^{(\beta)}(x;q)}{(q^{\alpha+1};q)_n} {}_2\phi_1 \left(\begin{matrix} q^{\alpha-\beta}, 0 \\ q^{\alpha+n+1} \end{matrix} ; q, t \right). \tag{37}$$

Proof. We start with the generating function for q-Laguerre polynomials found in [5, (14.21.14)]

$$\frac{1}{(t;q)_\infty} {}_0\phi_1 \left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix} ; q, -xtq^{\alpha+1} \right) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_n} t^n. \tag{38}$$

Using the connection relation (36) in (38), reversing the orders of the summations, shifting the n index by j, and using (4) through (11), obtains the desired result since $|a_n| \le |t|^n/(1 - q^{\alpha+1})^n$, $|c_{n,k}| \le K_9[n+1]^{\beta-\alpha+4}$, and

$$|L_n^{(\alpha)}(x;q)| \le [n+1]^{\sigma_8}/(1-q)^n, \tag{39}$$

implies

$$\sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} |c_{k,n}||L_k^{(\alpha)}(x;q)| \le K_9 \sum_{n=0}^{\infty} \frac{|t|^n}{(1-q^{\alpha+1})^n (1-q)^n} (n+1)^{\sigma_8} < \infty.$$

Therefore the theorem holds. □

Theorem 12. *Let $\alpha, \beta \in (-1, \infty)$, $0 < |q| < 1$, $|t| < (1 - q^{\alpha+1})(1 - q)$. Then*

$$(t;q)_\infty \, {}_0\phi_2 \left(\begin{matrix} - \\ q^{\alpha+1}, t \end{matrix} ; q, -xtq^{\alpha+1} \right) = \sum_{n=0}^{\infty} \frac{(-tq^{\alpha-\beta})^n q^{\binom{n}{2}} L_n^{(\beta)}(x;q)}{(q^{\alpha+1};q)_n} {}_1\phi_1 \left(\begin{matrix} q^{\alpha-\beta} \\ q^{\alpha+n+1} \end{matrix} ; q, tq^n \right). \tag{40}$$

Proof. We start with the generating function for the q-Laguerre polynomials found in [5, (14.21.15)]

$$(t;q)_\infty \, {}_0\phi_2 \left(\begin{matrix} - \\ q^{\alpha+1}, t \end{matrix} ; q, -xtq^{\alpha+1} \right) = \sum_{n=0}^{\infty} \frac{(-t)^n q^{\binom{n}{2}}}{(q^{\alpha+1};q)_n} L_n^{(\alpha)}(x;q). \tag{41}$$

Using the connection relation (36) in (41), reversing the orders of the summations, shifting the n index by j, and using (4) through (11), obtains the desired result since, again, $|a_n| \le |t|^n/(1 - q^{\alpha+1})^n$. □

Theorem 13. *Let $\alpha, \beta \in (-1, \infty)$, $\gamma \in \mathbb{C}$, $0 < |q| < 1$, $|t| < 1 - q$. Then*

$$\frac{(\gamma t;q)_\infty}{(t;q)_\infty} {}_1\phi_2 \left(\begin{matrix} \gamma \\ q^{\alpha+1}, \gamma t \end{matrix} ; q, -xtq^{\alpha+1} \right) = \sum_{n=0}^{\infty} \frac{(\gamma;q)_n (tq^{\alpha-\beta})^n}{(q^{\alpha+1};q)_n} L_n^{(\beta)}(x;q) \, {}_2\phi_1 \left(\begin{matrix} q^{\alpha-\beta}, \gamma q^n \\ q^{\alpha+n+1} \end{matrix} ; q, t \right). \tag{42}$$

Proof. We start with the generating function for the q-Laguerre polynomials found in [5, (14.21.15)]

$$\frac{(\gamma t;q)_\infty}{(t;q)_\infty} {}_1\phi_2 \left(\begin{matrix} \gamma \\ q^{\alpha+1}, \gamma t \end{matrix} ; q, -xtq^{\alpha+1} \right) = \sum_{n=0}^{\infty} \frac{(\gamma;q)_n}{(q^{\alpha+1};q)_n} L_n^{(\alpha)}(x;q) t^n. \tag{43}$$

Using the connection relation (36) in (43), reversing the orders of the summations, shifting the n index by j, and using (4) through (11), obtains the result

$$|a_n| \le \frac{|t|^n [n+1]^{|\gamma|+1}}{\alpha+1}.$$

Therefore the theorem holds. □

6. Definite integrals, infinite series, and q-integrals

Consider a sequence of orthogonal polynomials $(p_k(x;\boldsymbol{\alpha}))$ (over a domain A, with positive weight $w(x;\boldsymbol{\alpha})$) associated with a linear functional $\mathbf{u}$, where $\boldsymbol{\alpha}$ is a set of fixed parameters. Define s_k, $k \in \mathbb{N}_0$ by

$$s_k^2 := \int_A \{p_k(x;\boldsymbol{\alpha})\}^2 w(x;\boldsymbol{\alpha})\,\mathrm{d}x.$$

In order to justify interchange between a generalized generating function via connection relation and an orthogonality relation for p_k, we show that the double sum/integral converges in the L^2-sense with respect to the weight $w(x;\boldsymbol{\alpha})$. This requires

$$\sum_{k=0}^{\infty} d_k^2 s_k^2 < \infty, \tag{44}$$

where

$$d_k = \sum_{n=k}^{\infty} a_n c_{k,n}.$$

Here, a_n is the coefficient multiplying the orthogonal polynomial in the original generating function, and $c_{k,n}$ is the connection coefficient for p_k (with appropriate set of parameters).

Lemma 3. *Let $\mathbf{u}$ be a classical linear functional and let $(p_n(x))$, $n \in \mathbb{N}_0$ be the sequence of orthogonal polynomials associated with $\mathbf{u}$. If $|p_n(x)| \le K(n+1)^\sigma \gamma^n$, with K, σ and γ constants independent of n, then $|s_n| \le K(n+1)^\sigma \gamma^n |s_0|$.*

Proof. See [7, Lemma 6.1]. □

Given $|p_k(x;\boldsymbol{\alpha})| \le K(k+1)^\sigma \gamma^k$, with K, σ and γ constants independent of k, an orthogonality relation for p_k, and $|t| < 1/\gamma$, one has $\sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} |c_{k,n} s_k| < \infty$, which implies $\sum_{k=0}^{\infty} |d_k s_k| < \infty$. Therefore one has confirmed (44), indicating that we are justified in reversing the order of our generalized sums and the orthogonality relations under the above assumptions, which also are fulfilled for the polynomial families used throughout this paper.

In this section one has integral representations, infinite series, and representations in terms of the q-integral. In all the cases Lemma 3 can be applied and we are justified in interchanging the linear form and the infinite sum.

6.1. Definite integrals

6.1.1. The continuous q-ultraspherical/Rogers polynomials

The property of orthogonality for continuous q-ultraspherical/Rogers polynomials found in [5, (3.10.16)] is given by

$$\langle \mathbf{u}, C_m(x;\beta|q) C_n(x;\beta|q) \rangle = 2\pi \frac{(1-\beta)(\beta, q\beta; q)_\infty (\beta^2;q)_n}{(1-\beta q^n)(\beta^2, q; q)_\infty (q;q)_n} \delta_{m,n}, \tag{45}$$

where the linear functional $\mathbf{u}$ is defined in (19), and $w_R(x;\beta|q)$ is defined in (20). We will use this orthogonality relation for proofs of the following definite integrals.

Corollary 14. *Let $n \in \mathbb{N}_0$, $\beta, \gamma \in (-1,1) \setminus \{0\}$, $0 < |q| < 1$, $|t| < 1 - \beta^2$. Then*

$$\int_{-1}^{1} (te^{-i\theta};q)_\infty\, {}_2\phi_1\left(\begin{matrix} \beta, \beta e^{2i\theta} \\ \beta^2 \end{matrix}; q, te^{-i\theta}\right) C_n(x;\gamma|q) \frac{w_R(x;\gamma|q)}{\sqrt{1-x^2}}\,\mathrm{d}x$$
$$= 2\pi(-\beta t)^n \frac{q^{\binom{n}{2}} (\gamma, q\gamma; q)_\infty (\beta, \gamma^2; q)_n}{(\gamma^2, q; q)_\infty (q, \beta^2, q\gamma; q)_n}\, {}_2\phi_5\left(\begin{matrix} \beta\gamma^{-1}, \beta q^n \\ \gamma q^{n+1}, \pm\beta q^{n/2}, \pm\beta q^{(n+1)/2} \end{matrix}; q, \gamma(\beta t)^2 q^{2n+1}\right). \tag{46}$$

Proof. Using the generalized generating function (17) and the orthogonality relation (45), the proof follows as above. □

Corollary 15. *Let $n \in \mathbb{N}_0, \beta, \gamma \in (-1,1) \setminus \{0\}, 0 < |q| < 1, |t| < 1 - \beta^2$. Then*

$$\int_{-1}^{1} \frac{1}{(te^{i\theta};q)_\infty} {}_2\phi_1 \left(\begin{matrix} \beta, \beta e^{2i\theta} \\ \beta^2 \end{matrix} ; q, te^{-i\theta} \right) C_n(x;\gamma|q) \frac{w_R(x;\gamma|q)}{\sqrt{1-x^2}} \, dx$$
$$= 2\pi t^n \frac{(\gamma, q\gamma; q)_\infty (\beta, \gamma^2; q)_n}{(\gamma^2, q; q)_\infty (q, \beta^2, q\gamma; q)_n} {}_6\phi_5 \left(\begin{matrix} \beta\gamma^{-1}, \beta q^n, 0, 0, 0, 0 \\ \gamma q^{n+1}, \pm\beta q^{n/2}, \pm\beta q^{(n+1)/2} \end{matrix} ; q, \gamma t^2 \right). \tag{47}$$

Proof. Using the generalized generating function (22) and the orthogonality relation (45), the proof follows. □

Corollary 16. *Let $n \in \mathbb{N}_0, \gamma \in \mathbb{C}, \alpha, \beta \in (-1,1) \setminus \{0\}, 0 < |q| < 1, |t| < 1 - \beta^2$. Then*

$$\int_{-1}^{1} \frac{(\gamma te^{i\theta};q)_\infty}{(te^{i\theta};q)_\infty} {}_3\phi_2 \left(\begin{matrix} \gamma, \beta, \beta e^{2i\theta} \\ \beta^2, \gamma te^{i\theta} \end{matrix} ; q, te^{-i\theta} \right) C_n(x;\alpha|q) \frac{w_R(x;\alpha|q)}{\sqrt{1-x^2}} \, dx$$
$$= 2\pi t^n \frac{(\alpha, q\alpha; q)_\infty (\alpha^2, \gamma, \beta; q)_n}{(\alpha^2, q; q)_\infty (q, \beta^2, q\alpha; q)_n} {}_6\phi_5 \left(\begin{matrix} \beta/\alpha, \beta q^n, \pm(\gamma q^n)^{\frac{1}{2}}, \pm(\gamma q^{n+1})^{\frac{1}{2}} \\ \alpha q^{n+1}, \pm\beta q^{n/2}, \pm\beta q^{(n+1)/2} \end{matrix} ; q, \alpha t^2 \right). \tag{48}$$

Proof. Using the generalized generating function (24) and the orthogonality relation (45), the proof follows. □

Corollary 17. *Let $n \in \mathbb{N}_0, \beta, \gamma \in (-1,1) \setminus \{0\}, 0 < |q| < 1, |t| < \min\{(1-\beta^2)(1+\sqrt{q}|\beta|)(1-q|\gamma|), 1\}$. Then*

$$\int_{-1}^{1} {}_2\phi_1 \left(\begin{matrix} \pm\beta^{\frac{1}{2}} e^{i\theta} \\ -\beta \end{matrix} ; q, te^{-i\theta} \right) {}_2\phi_1 \left(\begin{matrix} \pm(q\beta)^{\frac{1}{2}} e^{-i\theta} \\ -q\beta \end{matrix} ; q, te^{i\theta} \right) C_n(x;\gamma|q) \frac{w_R(x;\gamma|q)}{\sqrt{1-x^2}} \, dx$$
$$= 2\pi t^n \frac{(\gamma, q\gamma; q)_\infty (\gamma^2, \beta, \pm\beta q^{\frac{1}{2}}; q)_n}{(\gamma^2, q; q)_\infty (\beta^2, -q\beta, q\gamma, q; q)_n}$$
$$\times {}_{10}\phi_9 \left(\begin{matrix} \beta\gamma^{-1}, \beta q^n, \pm(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \pm(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}}, \pm i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \pm i(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}} \\ \gamma q^{n+1}, \pm\beta q^{n/2}, \pm\beta q^{(n+1)/2}, \pm i(\beta q^{n+1})^{\frac{1}{2}}, \pm i(\beta q^{n+2})^{\frac{1}{2}} \end{matrix} ; q, \gamma t^2 \right). \tag{49}$$

Proof. Using the generalized generating function (26) and the orthogonality relation (45), the proof follows. □

Corollary 18. *Let $n \in \mathbb{N}_0, \beta, \gamma \in (-1,1) \setminus \{0\}, 0 < |q| < 1, |t| < \min\{(1-\beta^2)(1+\sqrt{q}|\beta|)(1-q|\gamma|), 1\}$. Then*

$$\int_{-1}^{1} {}_2\phi_1 \left(\begin{matrix} \beta^{\frac{1}{2}} e^{i\theta}, (q\beta)^{\frac{1}{2}} e^{i\theta} \\ \beta q^{\frac{1}{2}} \end{matrix} ; q, te^{-i\theta} \right) {}_2\phi_1 \left(\begin{matrix} -\beta^{\frac{1}{2}} e^{-i\theta}, -(q\beta)^{\frac{1}{2}} e^{-i\theta} \\ \beta q^{\frac{1}{2}} \end{matrix} ; q, te^{i\theta} \right) C_n(x;\gamma|q) \frac{w_R(x;\gamma|q)}{\sqrt{1-x^2}} \, dx$$
$$= 2\pi t^n \frac{(\gamma, q\gamma; q)_\infty (\gamma^2, \pm\beta, -\beta q^{\frac{1}{2}}; q)_n}{(\gamma^2, q; q)_\infty (\beta^2, \beta q^{\frac{1}{2}}, q\gamma, q; q)_n}$$
$$\times {}_{10}\phi_9 \left(\begin{matrix} \beta\gamma^{-1}, \beta q^n, \pm i(\beta q^n)^{\frac{1}{2}}, \pm i(\beta q^{n+1})^{\frac{1}{2}}, \pm i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \pm i(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}} \\ \gamma q^{n+1}, \pm\beta q^{n/2}, \pm\beta q^{(n+1)/2}, \pm(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \pm(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}} \end{matrix} ; q, \gamma t^2 \right). \tag{50}$$

Proof. Using the generalized generating function (28) and and the orthogonality relation (45), the proof follows. □

Corollary 19. *Let $n \in \mathbb{N}_0$ $\beta, \gamma \in (-1,1) \setminus \{0\}, 0 < |q| < 1, |t| < \min\{(1-\beta^2)(1+\sqrt{q}|\beta|), 1\}$. Then*

$$\int_{-1}^{1} {}_2\phi_1 \left(\begin{matrix} \beta^{\frac{1}{2}} e^{i\theta}, -(q\beta)^{\frac{1}{2}} e^{i\theta} \\ -\beta q^{\frac{1}{2}} \end{matrix} ; q, te^{-i\theta} \right) {}_2\phi_1 \left(\begin{matrix} (q\beta)^{\frac{1}{2}} e^{-i\theta}, -\beta^{\frac{1}{2}} e^{-i\theta} \\ -\beta q^{\frac{1}{2}} \end{matrix} ; q, te^{i\theta} \right) C_n(x;\gamma|q) \frac{w_R(x;\gamma|q)}{\sqrt{1-x^2}} \, dx$$
$$= \frac{2\pi t^n (\gamma, q\gamma; q)_\infty (\gamma^2, \pm\beta\beta q^{\frac{1}{2}}; q)_n}{(\gamma^2, q; q)_\infty (\beta^2, -\beta q^{\frac{1}{2}}, q\gamma, q; q)_n} {}_{10}\phi_9 \left(\begin{matrix} \beta\gamma^{-1}, \beta q^n, \pm i(\beta q^n)^{\frac{1}{2}}, \pm i(\beta q^{n+1})^{\frac{1}{2}}, \pm(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \pm(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}}, \\ \gamma q^{n+1}, \pm\beta q^{n/2}, \pm\beta q^{(n+1)/2}, \pm i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \pm i(\beta q^{n+\frac{3}{2}})^{\frac{1}{2}}, \end{matrix} ; q, \gamma t^2 \right). \tag{51}$$

Proof. Using the generalized generating function (30) and and the orthogonality relation (45), the proof follows. □

6.1.2. The q-Laguerre polynomials

The continuous orthogonality relation for the q-Laguerre polynomials is given by the following result. Notice that this result appears in [9, Section 2].

Proposition 1. *Let $\alpha \in (-1,\infty)$, $m,n \in \mathbb{N}_0$, $0<|q|<1$. Then*

$$\int_0^\infty L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)\frac{x^\alpha}{(-x;q)_\infty}\,\mathrm{d}x = -\frac{\delta_{m,n}}{q^n}\begin{cases}\dfrac{\pi(q^{-\alpha};q)_\infty(q^{\alpha+1};q)_n}{\sin(\pi\alpha)(q;q)_\infty(q;q)_n}, & \text{if } \alpha\in(-1,\infty)\setminus\mathbb{N}_0,\\[2ex] \dfrac{(q^{n+1};q)_\alpha\log q}{q^{\alpha(\alpha+1)/2}}, & \text{if } \alpha\in\mathbb{N}_0.\end{cases} \tag{52}$$

Proof. The continuous orthogonality relation for the q-Laguerre polynomials is given in [5, (14.21.2)] with the right-hand side expressed in terms of gamma functions, namely

$$\int_0^\infty L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)\frac{x^\alpha}{(-x;q)_\infty}\,\mathrm{d}x = \frac{(q^{-\alpha};q)_\infty(q^{\alpha+1};q)_n}{q^n(q;q)_\infty(q;q)_n}\Gamma(-\alpha)\Gamma(\alpha+1)\delta_{m,n}.$$

The gamma functions can be replaced using the reflection formula [8, (5.5.3)] and the result is given in the theorem for $\alpha\in(-1,\infty)\setminus\mathbb{N}_0$. The result for $\alpha\in\mathbb{N}_0$ is a consequence of (3) and [10, cf. (2.9)], namely

$$\lim_{\alpha\to k}\frac{(q^{1-\alpha};q)_\infty}{\sin(\pi\alpha)(aq^{-\alpha};q)_\infty} = \frac{-(q;q)_\infty(q;q)_{k-1}\log q}{\pi q^{\binom{k}{2}}(a;q)_\infty(a;q^{-1})_k},$$

which leads to

$$\lim_{\alpha\to k}\frac{(q^{-\alpha};q)_\infty}{\sin(\pi\alpha)} = \frac{(q;q)_\infty(q;q)_k\log q}{\pi q^{k(k+1)/2}}.$$

Applying this limit completes the proof. □

Corollary 20. *Let $n\in\mathbb{N}_0$, $\alpha,\beta\in(-1,\infty)$, $0<|q|<1$, $|t|<(1-q^{\alpha+1})(1-q)$. Then*

$$\begin{aligned}&\int_0^\infty {}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-xtq^{\alpha+1}\right)L_n^{(\beta)}(x;q)\frac{x^\beta}{(-x;q)_\infty}\,\mathrm{d}x\\ &= \frac{-\left(tq^{\alpha-\beta}\right)^n(t;q)_\infty}{q^n(q^{\alpha+1};q)_n}\,{}_2\phi_1\left(\begin{matrix}q^{\alpha-\beta},0\\ q^{\alpha+n+1}\end{matrix};q,t\right)\begin{cases}\dfrac{\pi(q^{-\beta};q)_\infty(q^{\beta+1};q)_n}{\sin(\pi\beta)(q;q)_\infty(q;q)_n}, & \text{if } \beta\in(-1,\infty)\setminus\mathbb{N}_0,\\[2ex] \dfrac{(q^{n+1};q)_\beta\log q}{q^{\beta(\beta+1)/2}}, & \text{if } \beta\in\mathbb{N}_0.\end{cases}\end{aligned} \tag{53}$$

Proof. Using the generalized generating function (37) and and the orthogonality relation (52), the proof follows. □

Corollary 21. *Let $n\in\mathbb{N}_0$, $\alpha,\beta\in(-1,\infty)$, $0<|q|<1$, $|t|<(1-q^{\alpha+1})(1-q)$. Then*

$$\begin{aligned}&\int_0^\infty {}_0\phi_2\left(\begin{matrix}-\\ q^{\alpha+1},t\end{matrix};q,-xtq^{\alpha+1}\right)L_n^{(\beta)}(x;q)\frac{x^\beta}{(-x;q)_\infty}\,\mathrm{d}x\\ &= \frac{-\left(-tq^{\alpha-\beta}\right)^n}{q^n(t;q)_\infty(q^{\alpha+1};q)_n}\,{}_1\phi_1\left(\begin{matrix}q^{\alpha-\beta}\\ q^{\alpha+n+1}\end{matrix};q,tq^n\right)\begin{cases}\dfrac{\pi(q^{-\beta};q)_\infty(q^{\beta+1};q)_n}{\sin(\pi\beta)(q;q)_\infty(q;q)_n}, & \text{if } \beta\in(-1,\infty)\setminus\mathbb{N}_0,\\[2ex] \dfrac{(q^{n+1};q)_\beta\log q}{q^{\beta(\beta+1)/2}}, & \text{if } \beta\in\mathbb{N}_0.\end{cases}\end{aligned} \tag{54}$$

Proof. Using the generalized generating function (40) and and the orthogonality relation (52), the proof follows. □

Corollary 22. *Let $n \in \mathbb{N}_0$, $\alpha, \beta \in (-1, \infty)$, $\gamma \in \mathbb{C}$, $0 < |q| < 1$, $|t| < 1 - q$. Then*

$$\int_0^\infty {}_1\phi_2\left(\begin{matrix}\gamma\\ q^{\alpha+1}, \gamma t\end{matrix}; q, -xtq^{\alpha+1}\right) L_n^{(\beta)}(x;q)\frac{x^\beta}{(-x;q)_\infty}\,\mathrm{d}x$$
$$= \frac{-\left(tq^{\alpha-\beta}\right)^n (t;q)_\infty(\gamma;q)_n}{q^n(\gamma t;q)_\infty(q^{\alpha+1};q)_n}\, {}_2\phi_1\left(\begin{matrix}q^{\alpha-\beta}, \gamma q^n\\ q^{\alpha+n+1}\end{matrix}; q, t\right)\begin{cases}\dfrac{\pi(q^{-\beta};q)_\infty(q^{\beta+1};q)_n}{\sin(\pi\beta)(q;q)_\infty(q;q)_n}, & \text{if } \beta \in (-1,\infty)\setminus\mathbb{N}_0,\\[2ex] \dfrac{(q^{n+1};q)_\beta \log q}{q^{\beta(\beta+1)/2}}, & \text{if } \beta \in \mathbb{N}_0.\end{cases} \tag{55}$$

Proof. Using the generalized generating function (42) and and the orthogonality relation (52), the proof follows. □

6.2. Infinite series

6.2.1. The little q-Laguerre/Wall polynomials

The little q-Laguerre/Wall polynomials satisfy a discrete orthogonality relation, namely [5, (14.20.2)]

$$\sum_{k=0}^\infty p_m(q^k;a|q)p_n(q^k;a|q)\frac{(aq)^k}{(q;q)_k} = \frac{(aq)^n(q;q)_n}{(aq;q)_\infty(aq;q)_n}\delta_{m,n}, \tag{56}$$

for $a \in (0, 1/q)$, with $0 < |q| < 1$.

Corollary 23. *Let $n \in \mathbb{N}_0$, $0 < |q| < 1$, $\alpha, \beta \in (0, q^{-1})$, $|t| < \min\{(1-\beta^2)(1+\sqrt{q}|\beta|), 1\}$. Then*

$$\sum_{k=0}^\infty \frac{(q\beta)^k}{(tq^k;q)_\infty}\, {}_0\phi_1\left(\begin{matrix}-\\ q\alpha\end{matrix}; q, t\alpha q^{k+1}\right)\frac{p_n\left(q^k;\beta|q\right)}{(q;q)_k} = \frac{q^{\binom{n}{2}}(-q\beta t)^n}{(t, q\beta;q)_\infty(q\alpha;q)_n}\, {}_1\phi_1\left(\begin{matrix}\alpha/\beta\\ \alpha q^{n+1}\end{matrix}; q, t\beta q^{n+1}\right). \tag{57}$$

Proof. We begin with the generalized generating function (33) and using the orthogonality relation (56) completes the proof. □

6.2.2. The q-Laguerre polynomials

One type of discrete orthogonality that the q-Laguerre polynomials satisfy is [5, (14.21.3)]

$$\sum_{k=-\infty}^\infty L_m^{(\alpha)}(cq^k;q)L_n^{(\alpha)}(cq^k;q)\frac{q^{(\alpha+1)k}}{(-cq^k;q)_\infty} = \frac{(q, -cq^{\alpha+1}, -q^{-\alpha}/c;q)_\infty(q^{\alpha+1};q)_n}{q^n(q^{\alpha+1}, -c, -q/c;q)_\infty(q;q)_n}\delta_{m,n}, \tag{58}$$

for $\alpha \in (-1, \infty)$, $c > 0$.

Corollary 24. *Let $n \in \mathbb{N}_0$, $0 < |q| < 1$, $\alpha, \beta \in (-1, \infty)$, $|t| < (1-q^{\alpha+1})(1-q)$, $c > 0$. Then*

$$\sum_{k=-\infty}^\infty {}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix}; q, -ctq^{\alpha+k+1}\right) L_n^{(\beta)}(cq^k;q)\frac{q^{(\beta+1)k}}{(-cq^k;q)_\infty}$$
$$= \frac{\left(tq^{\alpha-\beta}\right)^n (t, q, -cq^{\beta+1}, -q^{-\beta}/c;q)_\infty(q^{\beta+1};q)_n}{q^n(q^{\beta+1}, -c, -q/c;q)_\infty(q, q^{\alpha+1};q)_n}\, {}_2\phi_1\left(\begin{matrix}q^{\alpha-\beta}, 0\\ q^{\alpha+n+1}\end{matrix}; q, t\right). \tag{59}$$

Proof. We begin with the generalized generating function (37) and using the orthogonality relation (58) completes the proof. □

Corollary 25. *Let $n \in \mathbb{N}_0$, $0<|q|<1$, $\alpha,\beta \in (-1,\infty)$, $|t|<(1-q^{\alpha+1})(1-q)$, $c>0$. Then*

$$\sum_{k=-\infty}^{\infty} {}_0\phi_2\left(\begin{matrix} - \\ q^{\alpha+1}, t \end{matrix}; q, -ctq^{\alpha+k+1}\right) L_n^{(\beta)}(cq^k;q) \frac{q^{(\beta+1)k}}{(-cq^k;q)_\infty}$$
$$= \frac{\left(-tq^{\alpha-\beta}\right)^n q^{\binom{n}{2}} (q,-cq^{\beta+1},-q^{-\beta}/c;q)_\infty (q^{\beta+1};q)_n}{q^n (t,q^{\beta+1},-c,-q/c;q)_\infty (q,q^{\alpha+1};q)_n} {}_1\phi_1\left(\begin{matrix} q^{\alpha-\beta} \\ q^{\alpha+n+1} \end{matrix}; q, tq^n\right). \quad (60)$$

Proof. We begin with the generalized generating function (40) and using the orthogonality relation (58) completes the proof. □

Corollary 26. *Let $n \in \mathbb{N}_0$, $0<|q|<1$, $\alpha,\beta \in (-1,\infty)$, $\gamma \in \mathbb{C}$, $|t|<1-q$, $c>0$. Then*

$$\sum_{k=-\infty}^{\infty} {}_1\phi_2\left(\begin{matrix} \gamma \\ q^{\alpha+1}, \gamma t \end{matrix}; q, -ctq^{\alpha+k+1}\right) L_n^{(\beta)}(cq^k;q) \frac{q^{(\beta+1)k}}{(-cq^k;q)_\infty}$$
$$= \frac{\left(tq^{\alpha-\beta}\right)^n (t,q,-cq^{\beta+1},-q^{-\beta}/c;q)_\infty (\gamma,q^{\beta+1};q)_n}{q^n (\gamma t,q^{\beta+1},-c,-q/c;q)_\infty (q,q^{\alpha+1};q)_n} {}_2\phi_1\left(\begin{matrix} q^{\alpha-\beta}, \gamma q^n \\ q^{\alpha+n+1} \end{matrix}; q, t\right). \quad (61)$$

Proof. We begin with the generalized generating function (42) and using the orthogonality relation (58) completes the proof. □

6.3. The q-Integrals

6.3.1. The q-Laguerre polynomials

One type of orthogonality for the q-Laguerre polynomials is [5, (14.21.4)]

$$\int_0^\infty L_m^{(\alpha)}(x;q) L_n^{(\alpha)}(x;q) \frac{x^\alpha}{(-x;q)_\infty} \, d_q x = \frac{(1-q)(q,-q^{\alpha+1},-q^{-\alpha};q)_\infty (q^{\alpha+1};q)_n}{2q^n (q^{\alpha+1},-q,-q;q)_\infty (q;q)_n} \delta_{m,n}. \quad (62)$$

Using this orthogonality relation we can obtain new q-integrals using our generalized generating functions for the q-Laguerre polynomials.

Corollary 27. *Let $n \in \mathbb{N}_0$, $0<|q|<1$, $\alpha,\beta \in (-1,\infty)$, $|t|<(1-q^{\alpha+1})(1-q)$. Then*

$$\int_0^\infty {}_0\phi_1\left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix}; q, -xtq^{\alpha+1}\right) L_n^{(\beta)}(x;q) \frac{x^\beta}{(-x;q)_\infty} \, d_q x$$
$$= \frac{(1-q)\left(tq^{\alpha-\beta}\right)^n (t,q,-q^{\beta+1},-q^{-\beta};q)_\infty (q^{\beta+1};q)_n}{2q^n (q^{\beta+1},-q,-q;q)_\infty (q,q^{\alpha+1};q)_n} {}_2\phi_1\left(\begin{matrix} q^{\alpha-\beta}, 0 \\ q^{\alpha+n+1} \end{matrix}; q, t\right). \quad (63)$$

Proof. We begin with the generalized generating function (37) and using the orthogonality relation (62) completes the proof. □

Corollary 28. *Let $n \in \mathbb{N}_0$, $0<|q|<1$, $\alpha,\beta \in (-1,\infty)$, $|t|<(1-q^{\alpha+1})(1-q)$. Then*

$$\int_0^\infty {}_0\phi_2\left(\begin{matrix} - \\ q^{\alpha+1}, t \end{matrix}; q, -xtq^{\alpha+1}\right) L_n^{(\beta)}(x;q) \frac{x^\beta}{(-x;q)_\infty} \, d_q x$$
$$= \frac{(1-q)\left(-tq^{\alpha-\beta}\right)^n q^{\binom{n}{2}} (q,-q^{\beta+1},-q^{-\beta};q)_\infty (q^{\beta+1};q)_n}{2q^n (t,q^{\beta+1},-q,-q;q)_\infty (q,q^{\alpha+1};q)_n} {}_1\phi_1\left(\begin{matrix} q^{\alpha-\beta} \\ q^{\alpha+n+1} \end{matrix}; q, tq^n\right). \quad (64)$$

Proof. We begin with the generalized generating function (40) and using the orthogonality relation (62) completes the proof. □

Corollary 29. *Let $n \in \mathbb{N}_0$, $0 < |q| < 1$, $\alpha, \beta \in (-1, \infty)$, $\gamma \in \mathbb{C}$, $|t| < 1 - q$. Then*

$$\int_0^\infty {}_1\phi_2\left(\begin{matrix}\gamma\\ q^{\alpha+1}, \gamma t\end{matrix}; q, -xtq^{\alpha+1}\right) L_n^{(\beta)}(x;q) \frac{x^\beta}{(-x;q)_\infty} \mathrm{d}_q x$$
$$= \frac{(1-q)\left(tq^{\alpha-\beta}\right)^n (t, q, -q^{\beta+1}, -q^{-\beta}; q)_\infty (\gamma, q^{\beta+1}; q)_n}{2q^n (\gamma t, q^{\beta+1}, -q, -q; q)_\infty (q, q^{\alpha+1}; q)_n} {}_2\phi_1\left(\begin{matrix}q^{\alpha-\beta}, \gamma q^n\\ q^{\alpha+n+1}\end{matrix}; q, t\right). \quad (65)$$

Proof. We begin with the generalized generating function (42) and using the orthogonality relation (62) completes the proof. □

Author Contributions: All authors contributed equally in this paper. All authors read and approved the final version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Data Availability: All data required for this research is included within this paper.

Funding Information: R.S.C-S acknowledges financial support through the research project PGC2018-096504-B-C33 supported by Agencia Estatal de Investigación of Spain.

Acknowledgments: Much thanks to Hans Volkmer for valuable discussions.

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